

DIFFERENTIATION

1 Definition and differentiation rules

Exercise 1. Find the tangent line to each of the following curves at the indicated point:

a) $y = \frac{x}{x^2+1}$ at the origin,

c) $y = x^2 + 1$ at $(3, 10)$,

b) $y = \cos(x)$ at $(\pi/2, 0)$,

d) $y = |x|$ at $(1, 1)$.

Exercise 2. Calculate the derivative of each of the following functions:

a) $f(x) = \left(\sqrt[5]{x} - \frac{1}{\sqrt[5]{x}}\right)^5$,

f) $f(x) = \frac{1}{2}x|x|$,

b) $f(x) = \cos(\cos(\cos(x)))$,

g) $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$,

c) $f(x) = x^4 e^x \log(x)$,

h) $f(x) = \sin(x)\sqrt{\cos(x)}$,

d) $f(x) = x^x$,

i) $f(x) = \sqrt[4]{x^3 - \sqrt{1 - x^2}}$,

e) $f(x) = \sqrt{x}^{\sqrt{x}}$,

j) $f(x) = x^x + x^{1/x}$.

Exercise 3. Calculate the following limits:

a) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$,

c) $\lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h}$,

b) $\lim_{x \rightarrow 0} \frac{\sqrt[4]{1000+x} - \sqrt[4]{1000}}{x} \cdot \frac{1}{x+1}$,

d) $\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x}$.

Exercise 4. Verify that the function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \begin{cases} 2x, & \text{if } x < 0, \\ 3x^2, & \text{if } x \geq 0. \end{cases}$$

is continuous but not differentiable at the origin.

Exercise 5. Find the points at which the tangent line to the curve $y = 2x^3 - 3x^2 - 12x + 40$ is parallel to the x -axis.

2 The Mean Value Theorem

Exercise 6. Find the range of the function $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) = x^{1/x}$.

Exercise 7. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = e^{-x}(1 - e^{-2x})$. Determine:

a) the zeros of the function;

b) its critical points and its intervals of increase and decrease;

c) its relative extrema and intervals of convexity;

d) the range of the function.

Exercise 8. Consider the function $f: (-\infty, -1) \cup (0, +\infty) \rightarrow \mathbb{R}$ defined by $f(x) = \log\left(\frac{x+1}{x}\right) - \frac{1}{x}$. Find its range.

Exercise 9. Let $f: \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \arctan\left(\frac{1-x}{1+x}\right) + \arctan(x).$$

Find its range.

Exercise 10. Find the range of the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = e^{-x^2}(x^2 - 3)$.

Exercise 11. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ be defined by $f(x) = \arctan\left(\frac{1+x}{1-x}\right)$.

a) Study the continuity of f and its limits at $-\infty$ and $+\infty$.

b) Find the range of f .

Exercise 12. Prove that $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ for every $x \in [-1, 1]$.

Exercise 13. Prove that

$$\frac{x}{1+x^2} < \arctan(x) < x,$$

for every positive x .

Exercise 14. Verify that

$$\frac{x-1}{2} - \frac{(x-1)^2}{4} \leq \log(\sqrt{x})$$

for every $x \geq 1$.

Exercise 15. Verify the inequality $x - 1 \leq \log(x^x)$ for every $x > 0$.

Exercise 16.

a) Determine the number of zeros of $f(x) = 2x^3 - 9x^2 + 12x + 7$.

b) Calculate $f([1, 3])$.

Exercise 17. Determine the number of solutions of the equation $x + e^{-x} = 2$.

Exercise 18. Determine the number of solutions of $2x^2 = \cos(x) + x \sin(x)$.

Exercise 19.

a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 e^{|x|}$. Find its range.

b) Determine the number of solutions of the equation $f(x) = 7$.

Exercise 20. Let $f: \mathbb{R}^+ \setminus \{1/e\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{x \log^2(x)}{1 + \log(x)}.$$

a) Study the monotonicity and extrema of f , and find its range.

b) Discuss the number of solutions of the equation $f(x) = 0$.

3 L'Hôpital's rule

Exercise 21. Let $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{\log(1 - \sin(x)) - 2 \log(\cos(x))}{\sin(x)},$$

for $x \neq 0$, and let $f(0) = a$. Determine the value of a for which f is continuous at zero.

Exercise 22. Consider the function $g(x) = \log(x^2 - 2x + e^2 + 1)$, where $x \in \mathbb{R}$.

- a) Find the range of g .
- b) Calculate $\lim_{x \rightarrow +\infty} \frac{2g(x)}{x^2 + 1}$.
- c) Verify that the equation $\frac{2}{x^2 + 1} g(x) = 1$ has at least one positive solution.

Exercise 23. Calculate the following limits:

- a) $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4}$,
- b) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$,
- c) $\lim_{x \rightarrow \pi/2^-} \frac{2x - \pi}{\cos(x)}$,
- d) $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2}$,
- e) $\lim_{x \rightarrow +\infty} (5^x + 1)^{1/x}$,
- f) $\lim_{x \rightarrow +\infty} \frac{\log(2 + 3e^x)}{\sqrt{2 + 3x^2}}$.

Exercise 24. Calculate the following limits:

- a) $\lim_{x \rightarrow 0} \frac{\cos(x) + 3x - 1}{2x}$,
- b) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \cos(x)}{x \sin(2x)}$,
- c) $\lim_{x \rightarrow +\infty} \frac{\log(\log(x))}{\log(x)}$,
- d) $\lim_{x \rightarrow 0} \left(\frac{\cos(x) - \sin(x)}{1 + x^2} \right)^{\frac{\sin(x)}{x^2}}$.

Exercise 25. Calculate the following limits:

- a) $\lim_{x \rightarrow 0^+} (\cos(x) + 2 \sin(3x))^{\frac{1}{x}}$,
- b) $\lim_{x \rightarrow 0} \frac{(1 - \cos(x)) \sin(4x)}{x^3 \cos(\frac{\pi}{4} - x)}$,
- c) $\lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x - \cos(x)}$,
- d) $\lim_{x \rightarrow 1} \left(\tan\left(\frac{\pi x}{4}\right) \right)^{\tan(\frac{\pi x}{2})}$.

Exercise 26. Study the behaviour of the function $f: A \rightarrow \mathbb{R}$ at the point α in each of the following cases:

- a) $A = (2, +\infty)$, $f(x) = \frac{\sqrt{x} - \sqrt{2} + \sqrt{x - 2}}{\sqrt{x^2 - 4}}$, $\alpha = 2$.
- b) $A = \mathbb{R}^+ \setminus \{1\}$, $f(x) = \frac{1}{\log(x)} - \frac{1}{x - 1}$, $\alpha = 1$.
- c) $A = (1, +\infty)$, $f(x) = \frac{x^x - x}{1 - x - \log(x)}$, $\alpha = 1$.

Exercise 27. Calculate the following limits:

a) $\lim_{x \rightarrow 0} \left(\frac{e^x - \sin(x)}{1 + \sin^2(x)} \right)^{1/x^2},$

d) $\lim_{x \rightarrow \pi/2} \left(\frac{2 - 2 \sin(x)}{\cos(x)^2} \right)^{\tan(x)},$

b) $\lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) + x,$

e) $\lim_{x \rightarrow 0} (\log(1+x) + e^{-x})^{\sin(x)/x^2},$

c) $\lim_{x \rightarrow 0} \left(\frac{3 \sin(x) - 3x \cos(x)}{x^3} \right)^{\frac{1}{x}},$

f) $\lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 3x} \right)^{\frac{1}{\log(x+5)}}.$

4 Optimization

Exercise 28. Study the monotonicity, extrema and convexity of the following functions.

a) $y = 6 - 2x - x^2,$

b) $y = 3x^4 - 4x^3,$

c) $y = (x - 1)^3.$

Exercise 29. Find two positive numbers whose sum is 20 and whose product is as large as possible.

Exercise 30. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius r .

Exercise 31. Four congruent isosceles triangles are cut from a square of side 1. What is the maximum volume of the pyramid that can be formed? (Hint: the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base and h is its height.)

Exercise 32. Find the dimensions of the trapezoid of largest area that can be inscribed in a semicircle of radius 1.

Exercise 33. Find the dimensions of the rectangle of largest area that has one vertex at $(0, 0)$ and the opposite vertex in the first quadrant on the parabola $y = 6 - x^2$.

Exercise 34. Find the dimensions of the cross of largest area that is symmetric about both coordinate axes and can be inscribed in a circle of radius 1.

Exercise 35. Find the point (a, b) on the parabola $y = 3 - x^2$ such that the triangle determined by the tangent line at that point and the coordinate axes has minimum area.

5 Taylor polynomials

Exercise 36. Express the polynomial $x^4 - 5x^3 - 3x^2 + 7x + 6$ in powers of $(x - 2)$.

Exercise 37. Find an approximation to the real number α with error less than 10^{-2} in each of the following cases:

a) $\alpha = \sqrt{e},$

c) $\alpha = \cos(-1/2),$

e) $\alpha = \cos(1/3),$

b) $\alpha = \sin(1/2),$

d) $\alpha = \log(3/2),$

f) $\alpha = \sin(1/4).$

Exercise 38. Let f be a function whose third-order Taylor polynomial centred at 0 is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3}.$$

Find the third-order Taylor polynomial centred at 0 of the function $g(x) = xf(x)$.

Exercise 39. The second-order Taylor polynomial centred at the origin of a function f is $2 + x + 2x^2$. Find the Taylor polynomial of the same order and centre for $g(x) = xe^{f(x)-1}$.

Exercise 40. The second-order Taylor polynomial centred at the origin of a function f is

$$1 + \frac{x^2}{4}.$$

Find the second-order Taylor polynomial centred at 0 of the function $\log(1 + f(x))$.