Calculus 1

Lecture 1: The real line, intervals and inequalities

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What we will be able to do today

- Place the familiar number systems inside the real line.

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- ⋆ Read and combine intervals correctly.
- ⋆ Interpret absolute value as distance.
- ⋆ Solve absolute-value and rational inequalities.

The real line

The most common sets we will be working with

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$\mathbb{N}$	$1, 2, 3, \ldots$
$\mathbb{Z}$	$\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	numbers of the form p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$
$\mathbb{R} \setminus \mathbb{Q}$	irrational numbers, such as $\sqrt{2}$, π and e

Where does each number first appear?

Number	Smallest set in the chain
7	
-3	
$\frac{5}{8}$	
$\sqrt{2}$	

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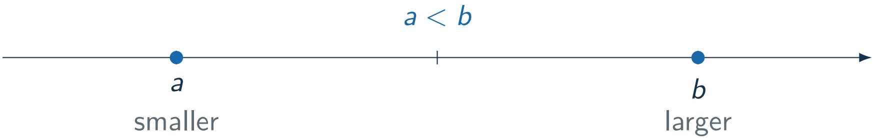
Where does each number first appear?

Number	Smallest set in the chain
7	$\mathbb{N}$
-3	$\mathbb{Z}$
$\dfrac{5}{8}$	$\mathbb{Q}$
$\sqrt{2}$	$\mathbb{R}$

Every number in the table is real.

Order gives the real line an orientation



Adding the same number preserves order:

$$a < b \implies a + c < b + c.$$

A negative factor reverses the inequality

$$-2x < 6$$

A negative factor reverses the inequality

$$-2x < 6$$

$$\frac{-2x}{-2} > \frac{6}{-2}$$

Dividing by -2 reverses the inequality.

A negative factor reverses the inequality

$$\begin{aligned} -2x &< 6 \\ \frac{-2x}{-2} &> \frac{6}{-2} \\ x &> -3. \end{aligned}$$

Dividing by -2 reverses the inequality.

A negative factor reverses the inequality

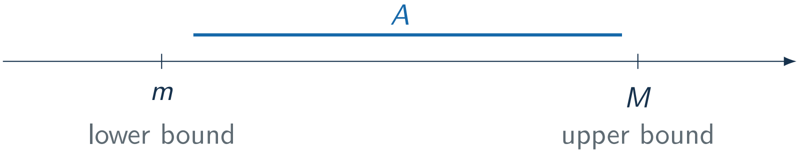
$$-2x < 6$$

$$\frac{-2x}{-2} > \frac{6}{-2}$$

$$x > -3.$$

Dividing by -2 reverses the inequality.

Bounds keep a set inside a finite region



$$\begin{aligned} m \le x \quad & \text{for every } x \in A, \\ x \le M \quad & \text{for every } x \in A. \end{aligned}$$

If both exist, A is **bounded**.

Supremum and infimum are the best bounds

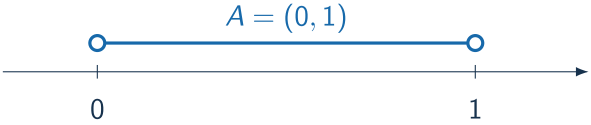
$$\sup A = \text{the least upper bound of } A,$$

$$\inf A = \text{the greatest lower bound of } A.$$

They need not belong to the set.

- If $\sup A \in A$, then $\max A = \sup A$.
- If $\inf A \in A$, then $\min A = \inf A$.

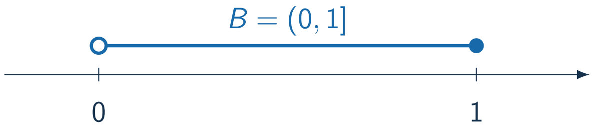
$A = (0,1)$ has bounds but no extrema



$$\inf A = 0, \qquad \min A \text{ does not exist,}$$
$$\sup A = 1, \qquad \max A \text{ does not exist.}$$

The endpoints are bounds, but neither endpoint belongs to A.

$B = (0, 1]$ attains its supremum



$$\inf B = 0, \qquad \min B \text{ does not exist,}$$
$$\sup B = 1, \qquad \max B = 1.$$

A filled endpoint belongs to the interval.

Completeness guarantees that the best bound exists

Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

This completeness property distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and supports the convergence theorems used later in Calculus.

Intervals

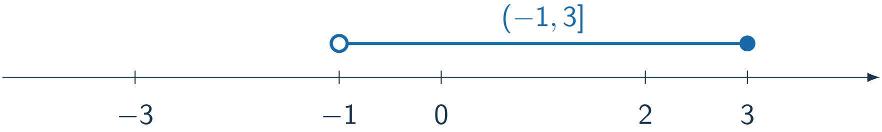
Brackets tell us whether endpoints are included

(a, b)	$a < x < b$
$[a, b]$	$a \leq x \leq b$
$(a, b]$	$a < x \leq b$
$[a, b)$	$a \leq x < b$

Round bracket: excluded

Square bracket: included

The interval $(-1, 3]$ has one open endpoint



-1 is excluded

$x > -1$

3 is included

$x \leq 3$

Infinity is never an included endpoint

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

$$(a, +\infty) = \{x \in \mathbb{R} : x > a\}$$

$+\infty$ and $-\infty$ are symbols, not real numbers.

Intersection means "both"; union means "at least one"

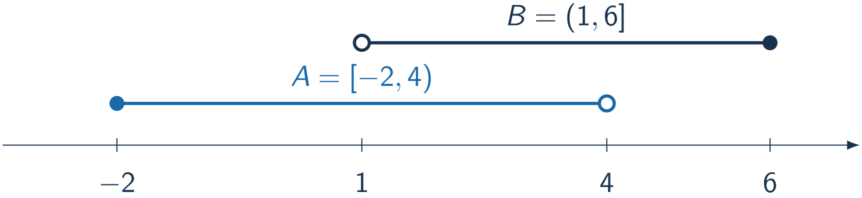
$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$\cap$ keeps only the overlap

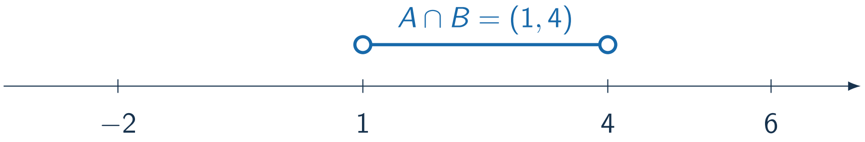
$\cup$ combines everything

$A = [-2, 4)$ and $B = (1, 6]$ overlap



First locate the common region; then check its endpoints.

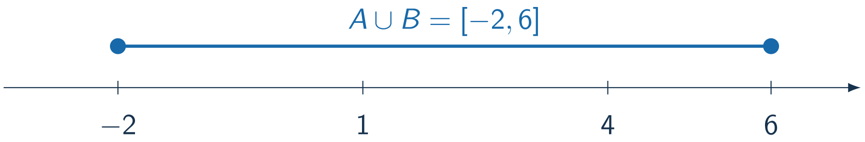
The intersection keeps only the overlap



- 1 is excluded because $1 \notin B$.
- 4 is excluded because $4 \notin A$.

$$A \cap B = (1, 4)$$

The union combines both intervals



There is no gap between the two intervals, and both outer endpoints are included.

$$A \cup B = [-2, 6]$$

Absolute value and distance

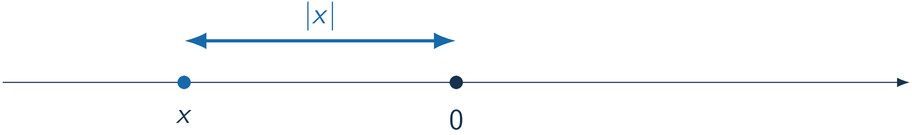
Absolute value removes the sign

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

$$|5| = 5 \qquad\qquad | - 5| = -(-5) = 5$$

$|x|$ is always non-negative.

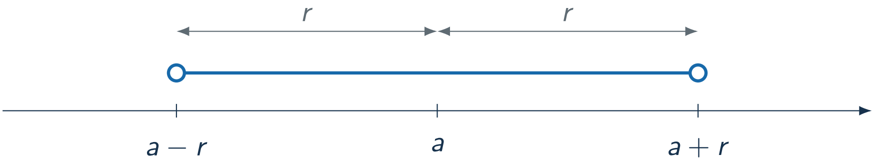
Geometrically, $|x|$ is the distance from x to 0



The distance between two numbers x and y is

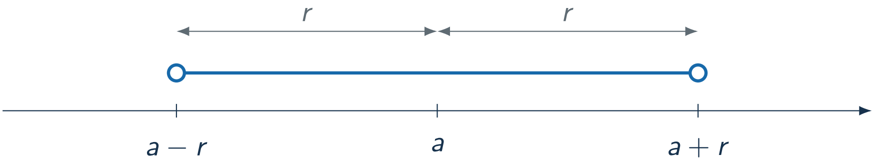
$$|x - y|.$$

$|x - a| < r$ describes the points close to a



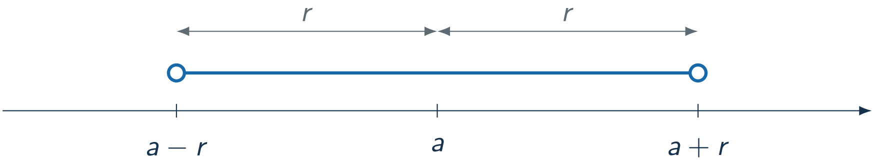
$$|x - a| < r$$

$|x-a|<r$ describes the points close to a



$$|x-a|<r \iff -r<x-a<r$$

$|x-a|<r$ describes the points close to a



$$|x-a|<r \iff -r<x-a<r \iff a-r<x<a+r.$$

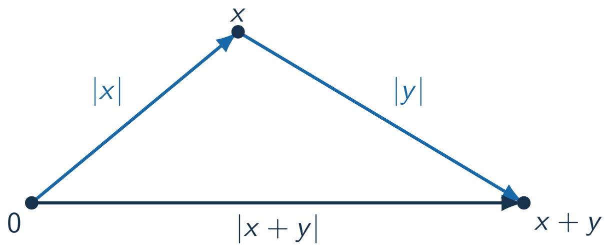
Four properties control absolute-value calculations

For all $x, y \in \mathbb{R}$,

$$
\begin{aligned}
|x| &\geq 0, \\
|x| &= 0 \iff x = 0, \\
|xy| &= |x|\,|y|, \\
|x+y| &\leq |x| + |y|, \\
\big||x| - |y|\big| &\leq |x-y|.
\end{aligned}
$$

The third statement is the triangle inequality.

A direct journey is never longer than a detour



$$|x+y| \le |x| + |y|$$

Example: solve $|2x - 3| < 5$

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Example: solve $|2x-3|<5$

$$
\begin{aligned}
|2x-3| &< 5 \\
-5 &< 2x-3 < 5
\end{aligned}
$$

Replace $|u|<r$ by $-r<u<r$.

Example: solve $|2x-3|<5$

$$\begin{aligned} &|2x-3|<5 \\ &\quad -5<2x-3<5 \\ &\quad -2<2x<8 \end{aligned}$$

Replace $|u|<r$ by $-r<u<r$.

Example: solve $|2x - 3| < 5$

$$\begin{aligned} &|2x-3| < 5 \\ &\quad -5 < 2x - 3 < 5 \\ &\quad -2 < 2x < 8 \\ &\quad -1 < x < 4. \end{aligned}$$

Replace $|u| < r$ by $-r < u < r$.

Example: solve $|2x - 3| < 5$

$$\begin{aligned} |2x - 3| &< 5 \\ -5 &< 2x - 3 < 5 \\ -2 &< 2x < 8 \\ -1 &< x < 4. \end{aligned}$$

Replace $|u| < r$ by $-r < u < r$.

$$x \in (-1, 4)$$

The solution $(-1, 4)$ is an open interval

The endpoints are excluded because the original inequality is strict.

Example: solve $|x-1| \geq 3$

$|x-1|$ is the distance from x to 1.

$$|x-1| \geq 3$$

$$x-1 \geq 3 \quad \text{or} \quad -(x-1) \geq 3$$

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$$x - 1 \leq -3 \quad \text{or} \quad x - 1 \geq 3$$

Example: solve $|x-1| \geq 3$

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$$\begin{aligned}
&|x-1| \geq 3 \\
&\quad x-1 \geq 3 \quad \text{or} \quad -(x-1) \geq 3 \\
&\quad x-1 \leq -3 \quad \text{or} \quad x-1 \geq 3 \\
&\qquad x \leq -2 \quad \text{or} \quad x \geq 4.
\end{aligned}$$

Example: solve $|x-1| \geq 3$

$|x-1|$ is the distance from x to 1.

$$\begin{aligned}
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x-1 \geq 3 \quad &\text{or} \quad -(x-1) \geq 3 \\
x-1 \leq -3 \quad &\text{or} \quad x-1 \geq 3 \\
x \leq -2 \quad &\text{or} \quad x \geq 4.
\end{aligned}$$

$$(-\infty, -2] \cup [4, +\infty)$$

Distance at least 3 means outside the central interval

$$(-\infty, -2] \cup [4, +\infty)$$

An approach to inequalities

Four checks prevent most inequality mistakes

1. Determine the domain of every expression.
2. Identify quantities whose sign may change.
3. Reverse the inequality after multiplying or dividing by a negative quantity.
4. Combine all conditions and check the endpoints.

Example: solve $\dfrac{x-1}{x+2} \geq 0$

The sign can change only when a factor is zero.

Numerator

$$x - 1 = 0 \implies x = 1$$

Denominator

$$x + 2 = 0 \implies x = -2$$

Example: solve $\dfrac{x-1}{x+2} \geq 0$

The sign can change only when a factor is zero.

Numerator

$$x - 1 = 0 \quad \Longrightarrow \quad x = 1$$

The quotient equals 0 here.

Denominator

$$x + 2 = 0 \quad \Longrightarrow \quad x = -2$$

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The quotient is not defined here.

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The quotient equals 0 here.

Denominator

$$x + 2 = 0 \quad \Longrightarrow \quad x = -2$$

The quotient is not defined here.

The critical points split the line into three intervals.

Test one point in each interval

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x-1$					
$x+2$					
$\dfrac{x-1}{x+2}$					

Test one point in each interval

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x-1$	$-$	$-$	$-$	0	$+$
$x+2$					
$\dfrac{x-1}{x+2}$					

Test one point in each interval

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x-1$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$
$\dfrac{x-1}{x+2}$					

Test one point in each interval

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x-1$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$
$\dfrac{x-1}{x+2}$	$+$	undefined	$-$	0	$+$

Test one point in each interval

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x-1$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$
$\dfrac{x-1}{x+2}$	$+$	undefined	$-$	0	$+$

We keep the places where the last row is non-negative.

Check the intervals and the critical points

- On $(-\infty, -2)$ the quotient is positive.
- On $(-2, 1)$ the quotient is negative.
- On $(1, +\infty)$ the quotient is positive.
- $x = -2$ is excluded because the quotient is undefined.
- $x = 1$ is included because the quotient equals 0.

$$(-\infty, -2) \cup [1, +\infty)$$

The solution appears directly on the real line

$$\frac{x-1}{x+2} \geq 0 \quad \Longleftrightarrow \quad x \in (-\infty, -2) \cup [1, +\infty).$$

What should remain from this lecture?

- ⋆ Supremum and infimum need not belong to the set.
- ⋆ Brackets encode whether interval endpoints are included.
- ⋆ Absolute value measures distance.
- ⋆ For inequalities: find the domain, locate sign changes and check endpoints.

A final check - This is for you, guys!

Without calculating on paper, decide:

1. Does $\sup(0, 2)$ belong to $(0, 2)$?
2. Is $-\infty$ an endpoint that can be included?
3. What geometric information is contained in $|x - 3| < 2$?
4. Which point must be excluded from $\dfrac{x-1}{x+2}$?

A final check - This is for you, guys!

Without calculating on paper, decide:

1. Does $\sup(0,2)$ belong to $(0,2)$?
2. Is $-\infty$ an endpoint that can be included?
3. What geometric information is contained in $|x-3|<2$?
4. Which point must be excluded from $\dfrac{x-1}{x+2}$?

Answers: no; no; x lies within distance 2 of 3; $x=-2$.