Calculus 1

Lecture 2: Functions and their graphs

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What we will be able to do today

- ⋆ Identify the domain, codomain and range of a function.

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- ⋆ Recognize the main elementary functions from their graphs.
- ⋆ Compose functions and determine when an inverse exists.

Functions and their domains

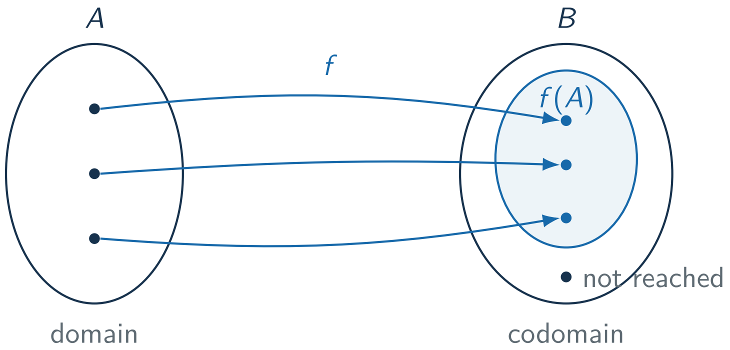
A function assigns exactly one output to each input

$$f : A \longrightarrow B$$

For every $x \in A$, there is **exactly one** value $f(x) \in B$.

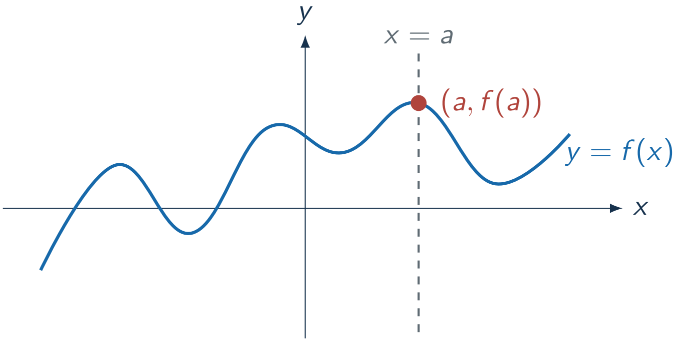
- A is the **domain**.
- B is the **codomain**.
- $f(A) = \{f(x) : x \in A\}$ is the **range** or **image**.

The range is the part of the codomain that is reached



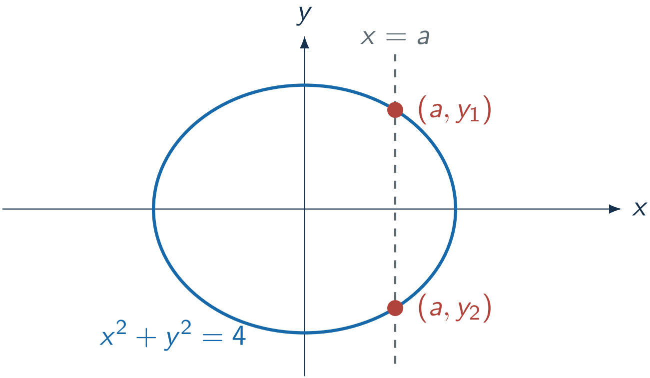
The range may be smaller than the codomain.

A graph must pass the vertical line test



Every vertical line meets the graph in **at most one point**.

A graph that fails the vertical line test



This vertical line meets the graph in **two points**. Therefore, the graph does not represent a function of x.

A formula determines its own natural domain

The natural domain contains all real numbers for which the formula makes sense.

- A denominator must be different from zero.
- The radicand of an even root must be non-negative.
- The argument of a logarithm must be positive.

Example: $f(x) = \sqrt{4 - x^2}$

The radicand must be non-negative:

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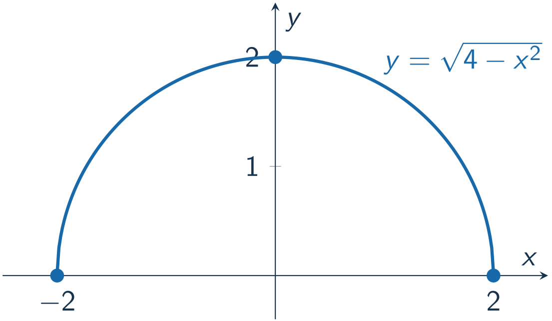
$$\mathrm{Dom}(f) = [-2, 2]$$

The graph is the upper half of a circle



$$\mathrm{Dom}(f) = [-2, 2], \qquad f([-2, 2]) = [0, 2].$$

Example: determine the domain

$$f(x) = \frac{\sqrt{x+1}}{\ln(3-x)}$$

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$$\begin{aligned} x+1 \geq 0 \quad &\Longrightarrow \quad x \geq -1, \\ 3-x > 0 \quad &\Longrightarrow \quad x < 3, \\ \ln(3-x) \neq 0 \quad &\Longrightarrow \quad 3-x \neq 1 \quad \Longrightarrow \quad x \neq 2. \end{aligned}$$

$$\mathrm{Dom}(f) = [-1, 2) \cup (2, 3)$$

Each restriction has a different source

Expression	Condition	Consequence
$\sqrt{x+1}$	$x+1 \geq 0$	$x \geq -1$
$\ln(3-x)$	$3-x>0$	$x<3$
denominator	$\ln(3-x) \neq 0$	$x \neq 2$

All three conditions must hold simultaneously:

$$[-1,+\infty) \cap (-\infty,3) \setminus \{2\} = [-1,2) \cup (2,3).$$

Elementary functions

Elementary functions come from a few basic families

- Polynomial and rational functions.

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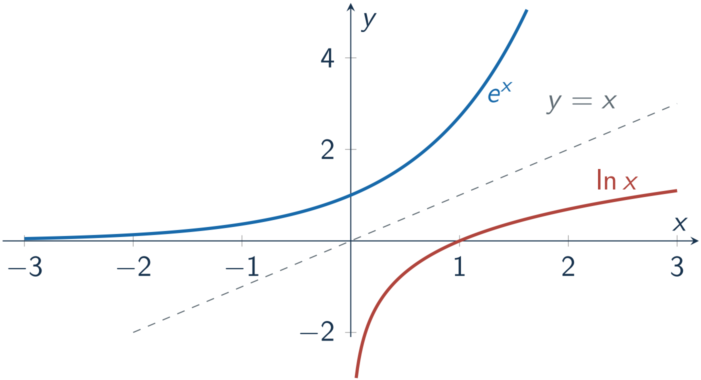
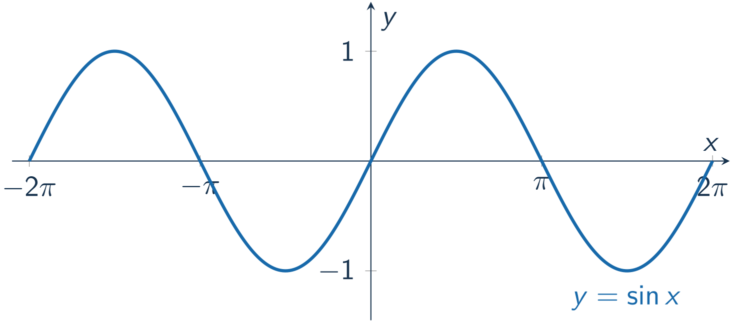
Sums, products, quotients and compositions build more complicated examples.

e^x and $\ln x$ are inverse functions



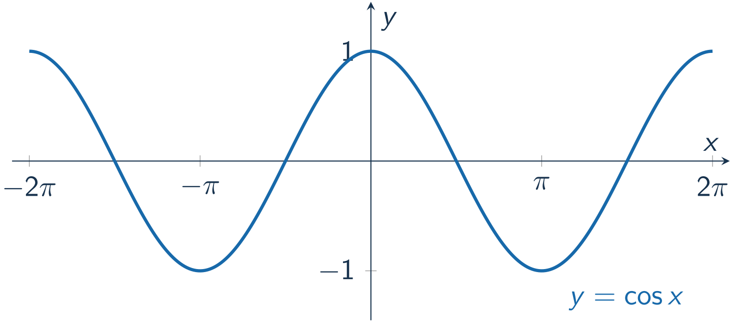
$e^x > 0$ on $\mathbb{R}$, while $\ln x$ is defined only for $x > 0$.

The sine function has range $[-1, 1]$ and period 2π



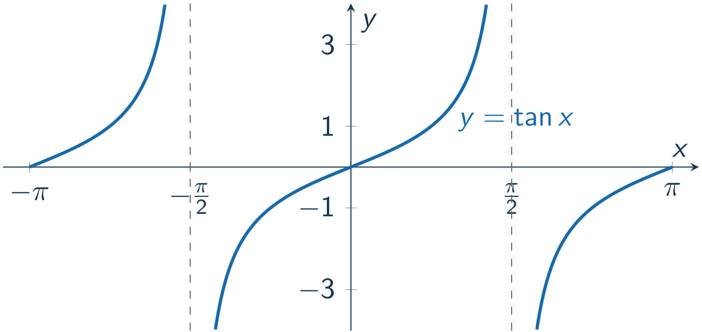
$$\sin(x + 2\pi) = \sin x, \qquad \sin(-x) = -\sin x.$$

The cosine function starts at its maximum



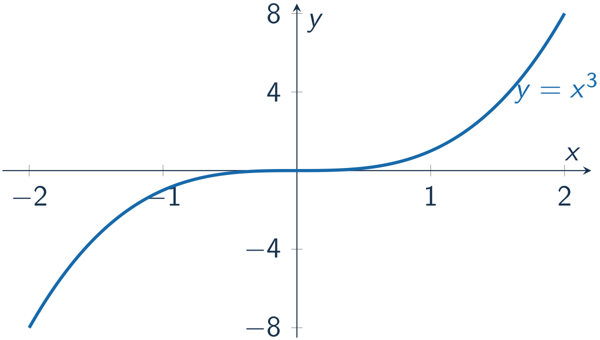
$$\cos(x + 2\pi) = \cos x, \qquad \cos(-x) = \cos x.$$

The tangent function has vertical asymptotes



The domain excludes $\frac{\pi}{2} + k\pi$, and the period is π.

x^3 is increasing on the whole real line



Its domain and range are both $\mathbb{R}$.

Composition and inverse functions

Composition

If $f : A \to B$ and $g : B \to C$, then

$$(g \circ f)(x) = g(f(x)).$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

The order matters: in general, $g \circ f \neq f \circ g$.

Example: first f, then g

Let

$$f(x) = x^2 + 1, \qquad g(x) = \sqrt{x}.$$

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$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) \\ &= g(x^2 + 1)\end{aligned}$$

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$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) \\ &= g(x^2 + 1) \\ &= \sqrt{x^2 + 1}.\end{aligned}$$

Since $x^2 + 1 > 0$ for every x, the domain is $\mathbb{R}$.

Example: reverse the order

With the same functions,

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With the same functions,

$$\begin{aligned}(f \circ g)(x) &= f\left(g(x)\right) \\ &= f(\sqrt{x}) \\ &= (\sqrt{x})^2 + 1 \\ &= |x| + 1 \\ &= x + 1.\end{aligned}$$

The square root requires $x \geq 0$, so the domain is $[0, +\infty)$.

The formula alone does not determine the function

$$(g \circ f)(x) = \sqrt{x^2 + 1}, \qquad x \in \mathbb{R},$$

$$(f \circ g)(x) = x + 1, \qquad x \in [0, +\infty).$$

Always record the domain together with the formula.

Bijectivity combines two different properties

For a function $f : A \to B$:

- ⋆ **Injective:** $f(x_1) = f(x_2) \implies x_1 = x_2$.

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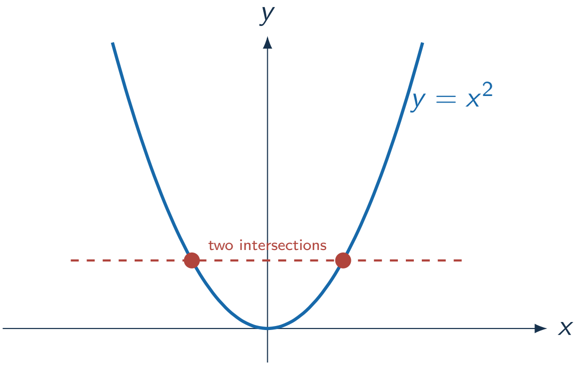
- ⋆ **Bijective:** f is both injective and surjective.

A bijective function has an inverse $f^{-1} : B \to A$.

Injectivity is checked with horizontal lines



The graph of x^2 fails the horizontal line test on $\mathbb{R}$.

x^2 is not injective on $\mathbb{R}$

$$f(1) = 1 \qquad \text{and} \qquad f(-1) = 1.$$

Thus,

$$f(1) = f(-1) \qquad \text{but} \qquad 1 \neq -1.$$

The same output is produced by two different inputs.

Restricting the domain makes x^2 bijective

Consider

$$f : [0, +\infty) \longrightarrow [0, +\infty), \qquad f(x) = x^2.$$

On this domain:

- f is strictly increasing, hence injective;
- every $y \geq 0$ equals $(\sqrt{y})^2$, hence f is surjective.

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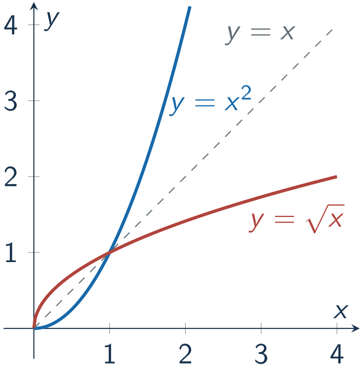
$$f^{-1}(x) = \sqrt{x}$$

A function and its inverse are reflected across $y = x$

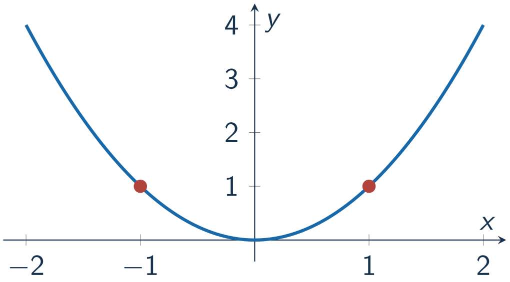


The coordinates (x, y) become (y, x).

Symmetry and periodicity

Even functions are symmetric about the y-axis

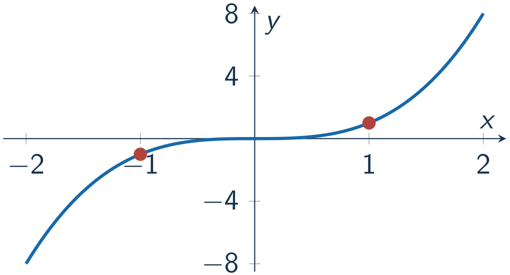
$$f(-x) = f(x).$$



x^2 and $\cos x$ are even.

Odd functions are symmetric about the origin

$$f(-x) = -f(x).$$



x^3 and $\sin x$ are odd.

A function may be neither even nor odd

For $f(x) = e^x$,

$$f(-x) = e^{-x}.$$

In general,

$$e^{-x} \neq e^x \qquad \text{and} \qquad e^{-x} \neq -e^x.$$

Therefore, e^x is neither even nor odd.

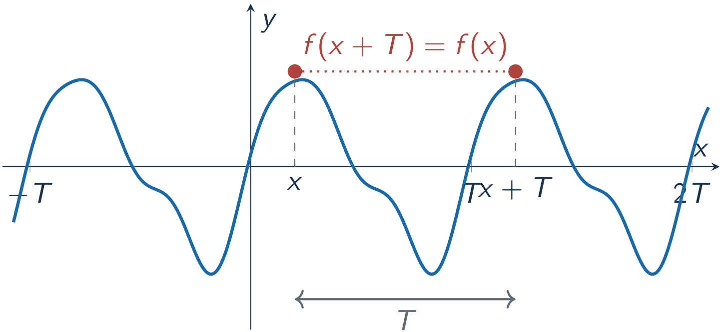
A periodic function repeats after a fixed length

A function $f : D \to \mathbb{R}$ is periodic if there is $T > 0$ such that

$$f(x+T) = f(x) \qquad (x \in D).$$

- $\sin x$ and $\cos x$ have period 2π.
- $\tan x$ has period π.

One period determines the whole graph



Graphs

Translations move a graph without changing its shape

Starting from $y = f(x)$:

$$y = f(x) + k \quad \Longrightarrow \quad \text{vertical shift by } k,$$

$$y = f(x - h) \quad \Longrightarrow \quad \text{horizontal shift by } h.$$

A positive h moves the graph to the right.

Scaling changes width or height

Assume $a \neq 0$ and $b \neq 0$.

$$y = af(x) \quad \Longrightarrow \quad \text{vertical factor } |a|,$$

$$y = f(bx) \quad \Longrightarrow \quad \text{horizontal factor } \frac{1}{|b|}.$$

If the factor is negative, the graph is also reflected.

Absolute values produce two useful transformations

$$y = |f(x)| \quad \Longrightarrow \quad \text{reflect the part below the } x\text{-axis upward,}$$

$$y = f(|x|) \quad \Longrightarrow \quad \text{copy the right half of the graph to the left.}$$

In the second transformation the result is always an even function.

Example: build $g(x) = -2|x-1| + 3$

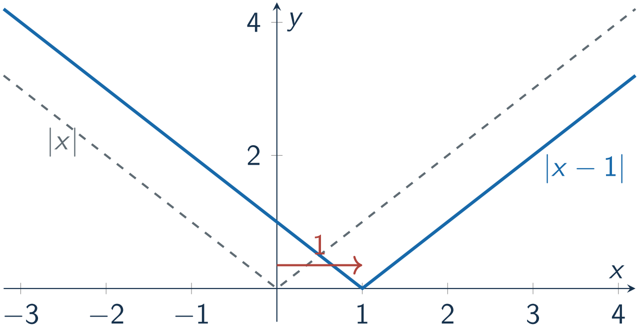
Begin with the familiar graph

$$f(x) = |x|.$$

The new expression records four operations:

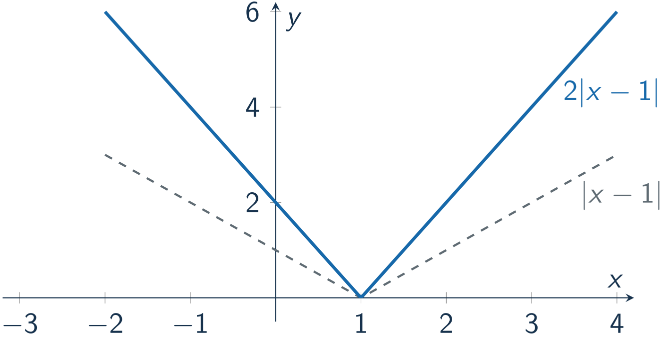
1. shift one unit to the right;
2. stretch vertically by a factor of 2;
3. reflect about the x-axis;
4. shift three units upward.

Step 1: shift one unit to the right



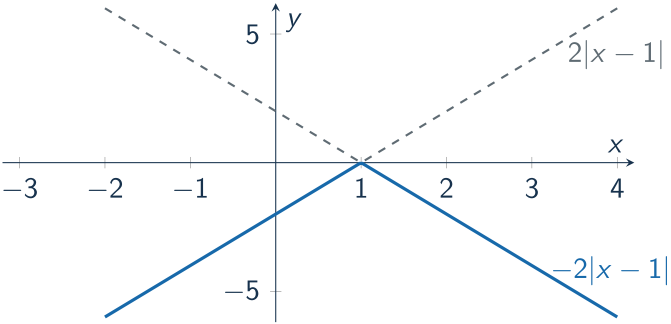
The vertex moves from $(0,0)$ to $(1,0)$.

Step 2: stretch vertically by a factor of 2



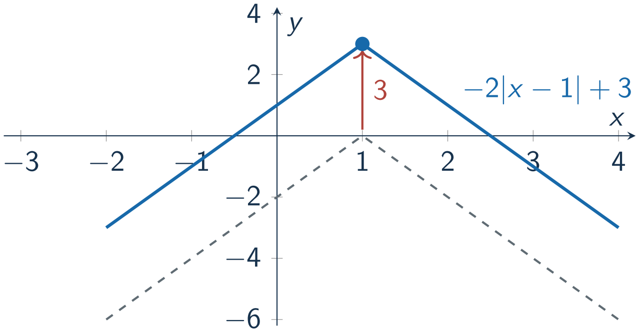
Every y-coordinate is multiplied by 2.

Step 3: reflect about the x-axis



The vertex remains at $(1, 0)$ and the graph now opens downward.

Step 4: shift three units upward



The final vertex is $(1, 3)$.

Find the x-intercepts of the transformed graph

$$-2|x-1|+3=0$$

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$$-2|x-1|+3=0$$

$$-2|x-1|=-3$$

Find the x-intercepts of the transformed graph

$$
\begin{aligned}
-2|x-1|+3 &= 0 \\
-2|x-1| &= -3 \\
|x-1| &= \frac{3}{2}
\end{aligned}
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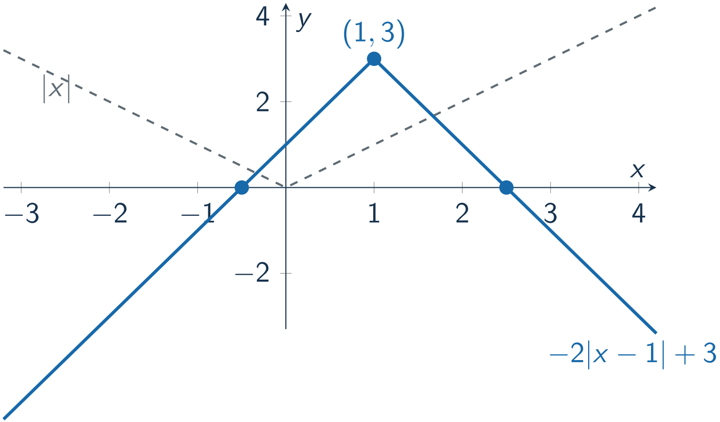
x-1 &= \pm\frac{3}{2} \\
x &= -\frac{1}{2} \quad \text{or} \quad x = \frac{5}{2}.
\end{aligned}
$$

The final graph records all four transformations



The x-intercepts are $-\frac{1}{2}$ and $\frac{5}{2}$.

What should remain from this lecture?

- A function gives exactly one output for every input in its domain.
- Domains come from all restrictions in the defining formula.
- Composition depends on order and on the available domains.
- Only bijective functions have inverses on their stated sets.
- Symmetry, periodicity and transformations help us read graphs quickly.

A final check

Without writing a full solution, decide:

1. What is the domain of $\sqrt{5-x}$?
2. Why is x^2 not invertible on $\mathbb{R}$?
3. Is $\cos x$ even, odd or neither?
4. In which direction does $f(x-3)$ move the graph of f?

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Answers: $(-\infty, 5]$; it is not injective; even; three units to the right.