Calculus 1

Practice 1: The Real Line and Elementary Functions

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Six problems connect the main ideas from Week 1

1. Absolute-value inequalities
2. Intervals and set operations
3. Natural domains
4. Compositions and their domains
5. Injectivity, range and inverse functions
6. Transformations of a cosine graph

Problem 1: Absolute-value inequalities

Absolute value measures distance

The statement

$$|x-a| \leq r$$

means that the distance from x to a is at most r.

For $r > 0$,

$$|x-a| \leq r \quad \Longleftrightarrow \quad a-r \leq x \leq a+r,$$

whereas

$$|x-a| > r \quad \Longleftrightarrow \quad x < a-r \ \text{ or } \ x > a+r.$$

Problem 1(a): place the expression between two bounds

Solve

$$|2x - 3| \leq 5.$$

Because $|u| \leq 5$ is equivalent to $-5 \leq u \leq 5$, we obtain

$$-5 \leq 2x - 3 \leq 5.$$

The same operation must be applied to all three parts.

Problem 1(a): isolate x in the double inequality

Start with

$$-5 \leq 2x - 3 \leq 5.$$

Add 3 throughout:

$$-2 \leq 2x \leq 8.$$

Divide throughout by 2:

$$-1 \leq x \leq 4.$$

$$x \in [-1, 4]$$

The endpoints are included because equality is allowed

Check the boundary values:

$$|2(-1) - 3| = 5, \qquad |2(4) - 3| = 5.$$

Problem 1(b): interpret the inequality as a distance

Solve

$$|x+1| > 2.$$

Since $|x+1| = |x-(-1)|$, the centre is -1.

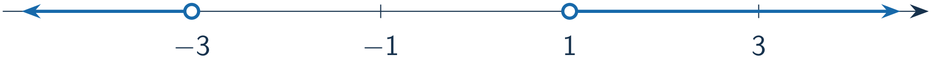
We want the points whose distance from -1 is greater than 2. The boundary points are

$$-1-2=-3, \qquad -1+2=1.$$

Problem 1(b): points outside the central interval work

The distance condition gives two possibilities:

$$x < -3 \qquad \text{or} \qquad x > 1.$$



$$x \in (-\infty, -3) \cup (1, +\infty)$$

Problem 1(c): compare two distances

Solve

$$|x - 2| \leq |x + 1| .$$

The left side is the distance from x to 2.
The right side is the distance from x to -1.

Problem 1(c): squaring is safe here

Both sides are non-negative, so

$$|x-2| \le |x+1| \iff (x-2)^2 \le (x+1)^2.$$

Expand:

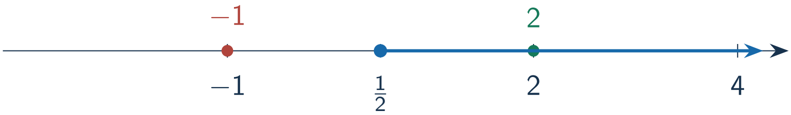
$$x^2 - 4x + 4 \le x^2 + 2x + 1.$$

Cancel x^2 and collect the terms:

$$-4x + 4 \le 2x + 1 \iff 3 \le 6x.$$

Problem 1(c): choose the side closer to 2

$$3 \leq 6x \quad \Longleftrightarrow \quad x \geq \frac{1}{2}.$$



$$x \in \left[\tfrac{1}{2}, +\infty\right)$$

Problem 2: Intervals and set operations

Begin by translating every set into interval notation

We are given

$$A = [-3, 2), \qquad B = (-1, 5], \qquad C = \{x \in \mathbb{R} : |x - 1| < 3\}.$$

For C,

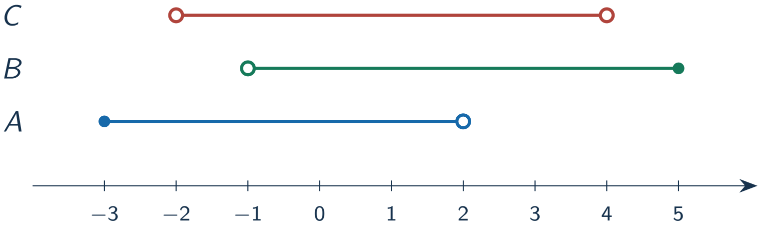
$$|x - 1| < 3 \iff -3 < x - 1 < 3.$$

Adding 1 throughout gives

$$-2 < x < 4.$$

$$C = (-2, 4)$$

Draw the three intervals on the same scale



Closed circles mean included endpoints; open circles mean excluded endpoints.

The intersection contains points belonging to both sets

For $A \cap B$, keep only the overlap of

$$A = [-3, 2) \qquad \text{and} \qquad B = (-1, 5].$$

The overlap starts after -1 and stops before 2.

Neither endpoint is included:

- $-1 \notin B$;
- $2 \notin A$.

$$A \cap B = (-1, 2)$$

The union contains points belonging to at least one set

The intervals $A = [-3, 2)$ and $B = (-1, 5]$ overlap, so their union has no gap.

It begins at -3, which belongs to A, and ends at 5, which belongs to B.

$$A \cup B = [-3, 5]$$

The internal endpoints -1 and 2 do not create holes because each belongs to the other interval.

Find $A \cap C$ before taking the difference

We have

$$A = [-3, 2), \qquad C = (-2, 4).$$

Their common part is

$$A \cap C = (-2, 2).$$

Now remove all points of

$$B = (-1, 5].$$

The point -1 is not contained in B, so it remains in the difference.

The set difference keeps the left-hand piece

$$(A \cap C) \setminus B = (-2, 2) \setminus (-1, 5].$$

All points larger than -1 are removed, while -1 itself is kept.

$$(A \cap C) \setminus B = (-2, -1]$$

The two endpoint decisions are different:

$$-2 \notin A \cap C, \qquad -1 \in A \cap C \text{ and } -1 \notin B.$$

Supremum and infimum need not belong to the set

Since

$$A \cap C = (-2, 2),$$

we have

$$\inf(A \cap C) = -2, \qquad \sup(A \cap C) = 2.$$

However,

$$-2 \notin A \cap C, \qquad 2 \notin A \cap C.$$

$A \cap C$ has neither a minimum nor a maximum.

Problem 3: Natural domains

Problem 3(a): exclude the zeros of the denominator

For

$$f(x) = \frac{x+1}{x^2-4},$$

we require

$$x^2 - 4 \neq 0.$$

Factor the denominator:

$$(x-2)(x+2) \neq 0.$$

Thus $x \neq 2$ and $x \neq -2$.

$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{-2, 2\}$$

Problem 3(b): the radicand must be non-negative

For

$$g(x) = \sqrt{\frac{x-1}{x+2}},$$

we need

$$\frac{x-1}{x+2} \geq 0, \qquad x \neq -2.$$

The critical points are

$$x = -2 \quad \text{and} \quad x = 1.$$

They divide the real line into three intervals.

Problem 3(b): complete the sign chart

x	$(-\infty, -2)$	$(-2, 1)$	$(1, +\infty)$
$x - 1$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$
$\dfrac{x-1}{x+2}$	$+$	$-$	$+$

At $x = 1$, the quotient is 0 and the square root is defined.
At $x = -2$, the quotient is not defined.

Problem 3(b): keep the non-negative intervals

The sign chart gives

$$\frac{x-1}{x+2} > 0 \quad \text{on} \quad (-\infty, -2) \cup (1, +\infty).$$

At $x = 1$, the quotient is zero, so this endpoint is included.
At $x = -2$, the denominator is zero, so this endpoint is excluded.

$$\mathrm{Dom}(g) = (-\infty, -2) \cup [1, +\infty)$$

Problem 3(c): a logarithm needs a positive argument

For

$$h(x) = \ln(4 - x^2),$$

we require

$$4 - x^2 > 0.$$

Equivalently,

$$x^2 < 4 \iff |x| < 2 \iff -2 < x < 2.$$

$$\mathrm{Dom}(h) = (-2, 2)$$

Problem 3(d): list every restriction separately

For

$$k(x) = \frac{\sqrt{x+3}}{\ln(x-1)},$$

we need all three conditions:

$$\begin{aligned} x+3 &\geq 0 && \text{from the square root,} \\ x-1 &> 0 && \text{from the logarithm,} \\ \ln(x-1) &\neq 0 && \text{from the denominator.} \end{aligned}$$

Problem 3(d): intersect the restrictions

The three conditions become

$$x \geq -3, \qquad x > 1, \qquad x - 1 \neq 1.$$

The last condition is $x \neq 2$.

The strongest interval condition is $x > 1$. Remove $x = 2$ from it:

$$x \in (1, +\infty) \setminus \{2\}.$$

$$\mathrm{Dom}(k) = (1, 2) \cup (2, +\infty)$$

Domain answers depend on strict and non-strict inequalities

Expression	Requirement	Endpoint rule
$1/q(x)$	$q(x) \neq 0$	exclude zeros
$\sqrt{q(x)}$	$q(x) \geq 0$	zero may be included
$\ln q(x)$	$q(x) > 0$	zero is excluded

The symbol $>$ or $\geq$ often decides whether an interval endpoint is open or closed.

Problem 4: Compositions

The order of composition matters

Let

$$f(x) = \sqrt{x-1}, \qquad g(x) = x^2 - 4.$$

The notation means

$$(f \circ g)(x) = f\big(g(x)\big), \qquad (g \circ f)(x) = g\big(f(x)\big).$$

For each composition, first form the formula and then determine where every intermediate expression is defined.

Problem 4: calculate $f \circ g$

Insert $g(x) = x^2 - 4$ into f:

$$\begin{aligned}(f \circ g)(x) &= f(x^2 - 4) \\ &= \sqrt{(x^2 - 4) - 1} \\ &= \sqrt{x^2 - 5}.\end{aligned}$$

The square root requires

$$x^2 - 5 \geq 0.$$

Problem 4: solve the domain condition for $f \circ g$

$$x^2 - 5 \geq 0 \quad \Longleftrightarrow \quad x^2 \geq 5 \quad \Longleftrightarrow \quad |x| \geq \sqrt{5}.$$

Therefore,

$$x \leq -\sqrt{5} \qquad \text{or} \qquad x \geq \sqrt{5}.$$

$$(f \circ g)(x) = \sqrt{x^2 - 5},$$
$$\mathrm{Dom}(f \circ g) = (-\infty, -\sqrt{5}] \cup [\sqrt{5}, +\infty)$$

Problem 4: calculate $g \circ f$

Insert $f(x) = \sqrt{x-1}$ into g:

$$\begin{aligned}(g \circ f)(x) &= g\left(\sqrt{x-1}\right) \\ &= \left(\sqrt{x-1}\right)^2 - 4 \\ &= x - 5.\end{aligned}$$

The simplified formula is not enough to determine the domain.

Problem 4: preserve the domain of the inner function

Before simplification, the composition contains

$$\sqrt{x-1}.$$

Therefore, we must still require

$$x-1 \geq 0 \quad \Longleftrightarrow \quad x \geq 1.$$

$$(g \circ f)(x) = x-5, \quad \mathrm{Dom}(g \circ f) = [1, +\infty)$$

Simplifying a formula never restores points that were excluded before the simplification.

Problem 5: Inverse functions

The function has one forbidden input

Consider

$$f(x) = \frac{3x-2}{x+4}, \qquad x \neq -4.$$

We will answer three questions:

1. Is f one-to-one?
2. Which real values occur as outputs?
3. What formula reverses f?

Problem 5: begin the injectivity test

Assume that two inputs have the same output:

$$f(x_1) = f(x_2).$$

Then

$$\frac{3x_1 - 2}{x_1 + 4} = \frac{3x_2 - 2}{x_2 + 4}.$$

Because $x_1, x_2 \neq -4$, both denominators are nonzero and we may cross-multiply.

Problem 5: equal outputs force equal inputs

Cross-multiplication gives

$$(3x_1 - 2)(x_2 + 4) = (3x_2 - 2)(x_1 + 4).$$

Expand both sides:

$$3x_1x_2 + 12x_1 - 2x_2 - 8 = 3x_1x_2 + 12x_2 - 2x_1 - 8.$$

Cancel equal terms and collect the rest:

$$14x_1 = 14x_2.$$

Hence $x_1 = x_2$.

f is injective.

Problem 5: solve the equation $y = f(x)$ for x

Start with

$$y = \frac{3x - 2}{x + 4}.$$

Multiply by $x + 4$:

$$y(x + 4) = 3x - 2.$$

Expand and collect the terms containing x:

$$yx + 4y = 3x - 2 \quad \Longleftrightarrow \quad x(y - 3) = -2 - 4y.$$

Problem 5: isolate x

If $y \neq 3$, then

$$x = \frac{-2-4y}{y-3} = \frac{2+4y}{3-y}.$$

This produces a real input for every $y \neq 3$.
Moreover, this input cannot be -4, because

$$\frac{2+4y}{3-y} = -4$$

would imply $2 = -12$.

Problem 5: the output 3 is impossible

If $f(x) = 3$, then

$$\frac{3x-2}{x+4} = 3.$$

Cross-multiplication gives

$$3x - 2 = 3x + 12,$$

which is impossible.

Thus 3 is not in the range, while every $y \neq 3$ is attained.

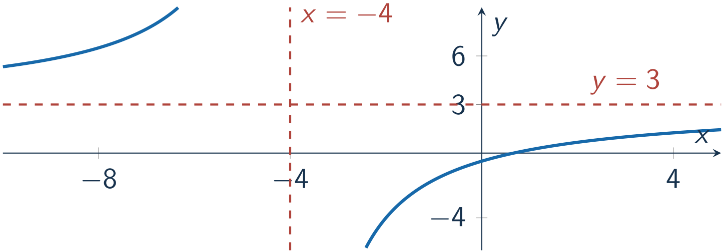
$$f(\mathbb{R} \setminus \{-4\}) = \mathbb{R} \setminus \{3\}$$

Problem 5: interchange the variables to write the inverse

We found
$$x = \frac{2+4y}{3-y}.$$
Interchanging x and y gives
$$f^{-1}(x) = \frac{2+4x}{3-x}.$$

$$f^{-1} : \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R} \setminus \{-4\}, \quad f^{-1}(x) = \frac{2+4x}{3-x}$$

The graph shows the excluded input and output



The vertical asymptote corresponds to the excluded input; the horizontal asymptote corresponds to the excluded output.