Calculus 1

Week 1

Problem 1. Let
$$M = \left\{ \frac{m-1}{m} : m \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}.$$
Find $\inf M$, $\sup M$, $\min M$ and $\max M$, whenever the latter two exist.

Solution. For every $m \in \mathbb{N}$,
$$\frac{m-1}{m} = 1 - \frac{1}{m}.$$
Consequently,
$$0 \leqslant \frac{m-1}{m} < 1.$$
Thus, 0 is a lower bound and 1 is an upper bound. Since the value corresponding to $m = 1$ is 0, we have
$$\inf M = \min M = 0.$$

The elements of M approach 1 from below. Therefore, no number smaller than 1 is an upper bound, and hence
$$\sup M = 1.$$
Since $1 \notin M$, the set M has no maximum.

Problem 2. Let
$$M = \{x \in \mathbb{Q} : x^2 < 2\}.$$
Find $\inf M$, $\sup M$, $\min M$ and $\max M$, whenever the latter two exist.

Solution. The inequality $x^2 < 2$ is equivalent to
$$-\sqrt{2} < x < \sqrt{2}.$$
Therefore, $-\sqrt{2}$ is a lower bound and $\sqrt{2}$ is an upper bound for M. Since there are elements of M arbitrarily close to $\sqrt{2}$ from the left and arbitrarily close to $-\sqrt{2}$ from the right, it follows that
$$\inf M = -\sqrt{2}, \qquad \sup M = \sqrt{2}.$$
Neither $\sqrt{2}$ nor $-\sqrt{2}$ is rational. In particular, neither belongs to M. Consequently, M has neither a maximum nor a minimum.

Problem 3. Use the definition of absolute value to calculate $|-3|$, $|0|$ and $|3|$.

Solution. Recall that

$$|x| = \begin{cases} -x, & x < 0, \\ x, & x \geqslant 0. \end{cases}$$

Therefore, $|-3| = 3$, $|0| = 0$ and $|3| = 3$.

Problem 4. Solve the inequality

$$\frac{x^2 + x + 1}{x^2 - 1} \geqslant 0.$$

Solution. We first study the numerator. Its discriminant is

$$\Delta = 1 - 4 = -3 < 0.$$

Since the leading coefficient is positive, we have

$$x^2 + x + 1 > 0$$

for every $x \in \mathbb{R}$. Therefore, the sign of the quotient is determined entirely by the denominator

$$x^2 - 1 = (x - 1)(x + 1).$$

The critical points are -1 and 1, and the sign chart is

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$x^2 + x + 1$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$
$\dfrac{x^2 + x + 1}{x^2 - 1}$	$+$	$-$	$+$

The points -1 and 1 are excluded because the expression is not defined there. Hence, the solution is

$$(-\infty, -1) \cup (1, +\infty).$$

Problem 5. Solve the inequality

$$|2x - 4| > 3.$$

Solution. The inequality $|u| > a$, where $a > 0$, is equivalent to

$$u < -a \qquad \text{or} \qquad u > a.$$

Thus,

$$2x - 4 < -3 \qquad \text{or} \qquad 2x - 4 > 3.$$

Solving the two inequalities gives

$$x < \frac{1}{2} \qquad \text{or} \qquad x > \frac{7}{2}.$$

Therefore, the solution set is

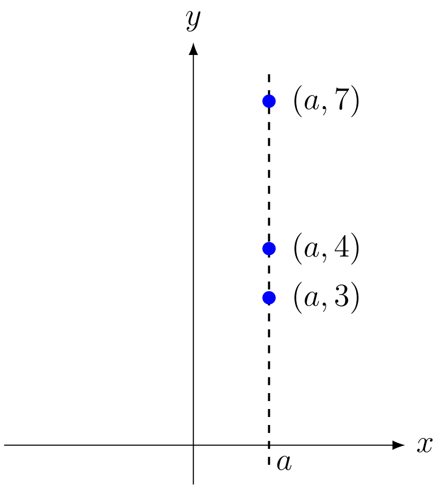
$$\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{7}{2}, +\infty\right).$$

Problem 6. Consider the relation

$$G = \{(a,3),(a,4),(a,7)\}.$$

Can G be the graph of a function of x?

Solution. No. The same input a is paired with the three different outputs 3, 4 and 7. A function must assign exactly one output to each element of its domain. Equivalently, the vertical line $x = a$ meets the relation in three points, so the relation fails the vertical line test.



Problem 7. Consider the functions $f : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$

$$f(x) = \frac{x^2 - 1}{x - 1}$$

and $g : \mathbb{R} \longrightarrow \mathbb{R}$

$$g(x) = x + 1.$$

Are f and g equal? Describe their graphs.

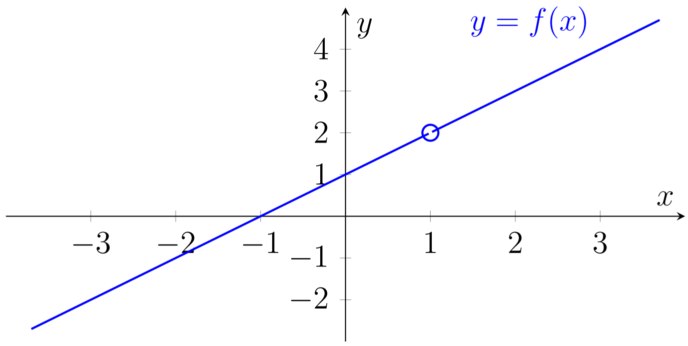
Solution. For every $x \neq 1$,

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 = g(x).$$

However, the domains are different:

$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{1\}, \qquad \mathrm{Dom}(g) = \mathbb{R}.$$

Consequently, $f \neq g$. The graph of g is the line $y = x + 1$. The graph of f is the same line with the point $(1, 2)$ removed.

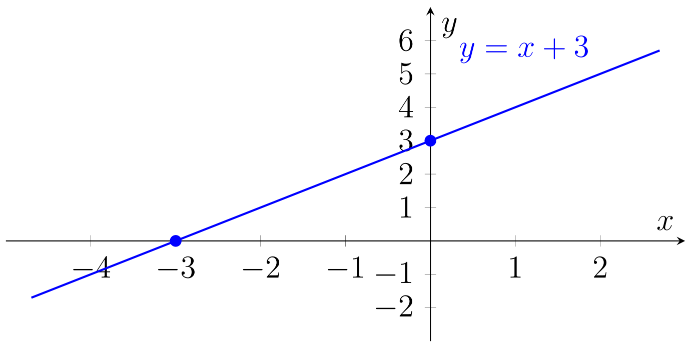


Problem 8. Sketch the graph of

$$f(x) = x + 3.$$

Find its intercepts, domain and range.

Solution. The graph is a straight line with slope 1. Since $f(0) = 3$, the y-intercept is $(0, 3)$. To find the x-intercept, we solve $x + 3 = 0$, which gives $x = -3$. Thus, the x-intercept is $(-3, 0)$. Moreover, $\mathrm{Dom}(f) = \mathbb{R}$ and $f(\mathbb{R}) = \mathbb{R}$.



Problem 9. Sketch the graph of

$$f(x) = x^2 + 4x + 4.$$

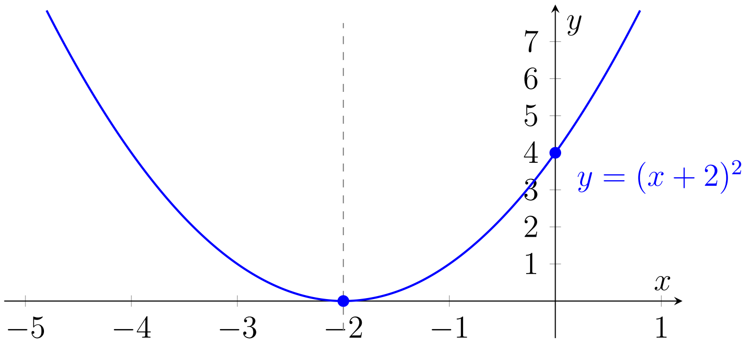
Find its vertex, axis of symmetry, intercepts, domain and range.

Solution. Completing the square gives

$$f(x) = x^2 + 4x + 4 = (x + 2)^2.$$

Hence, the graph is the parabola $y = x^2$ shifted two units to the left. Its vertex is $(-2, 0)$, and its axis of symmetry is $x = -2$. The only x-intercept is $(-2, 0)$, while the y-intercept is $(0, 4)$. Finally,

$$\mathrm{Dom}(f) = \mathbb{R}, \qquad f(\mathbb{R}) = [0, +\infty).$$



Problem 10. For each pair of functions, show that the range of f is contained in the domain of g, and then calculate $g \circ f$, including its natural domain.

(a) $f(x) = -\sqrt{x}$ and $g(x) = \sqrt{2-x}$;

(b) $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{x+1}{x-1}$.

Solution. **(a)** We have $\mathrm{Dom}(f) = [0, +\infty)$ and $f\big([0,+\infty)\big) = (-\infty, 0]$. On the other hand, the square root in g requires $2 - x \geqslant 0$, so $\mathrm{Dom}(g) = (-\infty, 2]$. Consequently, the range of f is contained in the domain of g. For $x \geqslant 0$,

$$(g \circ f)(x) = g\big(-\sqrt{x}\big) = \sqrt{2 + \sqrt{x}}.$$

Therefore, $\mathrm{Dom}(g \circ f) = [0, +\infty)$.

(b) First, $\mathrm{Dom}(f) = \mathbb{R} \setminus \{-1\}$. The value 1 does not belong to the range of f, because the equation

$$\frac{x}{x+1} = 1$$

has no solution. Conversely, if $y \neq 1$, then

$$y = \frac{x}{x+1} \quad \Longleftrightarrow \quad x = \frac{y}{1-y}.$$

Hence, $f\big(\mathbb{R} \setminus \{-1\}\big) = \mathbb{R} \setminus \{1\}$. Since $\mathrm{Dom}(g) = \mathbb{R} \setminus \{1\}$, the range of f is precisely the domain of g. Finally,

$$\begin{aligned}(g \circ f)(x) &= \frac{f(x)+1}{f(x)-1} \\ &= \frac{\dfrac{x}{x+1} + 1}{\dfrac{x}{x+1} - 1} = \frac{\dfrac{2x+1}{x+1}}{\dfrac{-1}{x+1}} = -2x - 1.\end{aligned}$$

The original function f is not defined at $x = -1$, and therefore $\mathrm{Dom}(g \circ f) = \mathbb{R} \setminus \{-1\}$.

Problem 11. Let $f(x) = \sin x$ and $g(x) = x^2$. Calculate $f \circ g$ and $g \circ f$. Are the two compositions equal?

Solution. Both functions are defined on $\mathbb{R}$, and the range of each function is contained in the domain of the other. Thus, both compositions have domain $\mathbb{R}$. We obtain

$$(f \circ g)(x) = f(x^2) = \sin(x^2)$$

and

$$(g \circ f)(x) = g(\sin x) = (\sin x)^2 = \sin^2 x.$$

These functions are not equal. For instance,

$$(f \circ g)(1) = \sin 1 \quad \text{and} \quad (g \circ f)(1) = \sin^2 1,$$

and $0 < \sin 1 < 1$, so $\sin 1 \neq \sin^2 1$. Therefore, in general, the composition of functions is not commutative.

Problem 12. Determine whether each function is bounded above, bounded below or bounded. Give an upper or lower bound whenever one exists.

(a) $f(x) = \sin x$.

(b) $g(x) = x^2$.

Solution. **(a)** Since $-1 \leqslant \sin x \leqslant 1$ for every $x \in \mathbb{R}$, the function f is bounded above by 1 and bounded below by -1. Therefore, f is bounded and $f(\mathbb{R}) = [-1, 1]$.

(b) We have $x^2 \geqslant 0$ for every $x \in \mathbb{R}$, so g is bounded below by 0. It is not bounded above. Consequently, g is bounded below but is not bounded.

Problem 13. Decide whether each of the following functions is even, odd or neither:

(a) $f(x) = x^3$.

(b) $g(x) = x^2$.

(c) $h(x) = 2x - x^2$.

(d) $k(x) = |x|$.

Solution. All four functions have the symmetric domain $\mathbb{R}$.

(a) For every $x \in \mathbb{R}$,

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Hence, f is odd.

(b) We have

$$g(-x) = (-x)^2 = x^2 = g(x),$$

so g is even.

(c) In this case,

$$h(-x) = -2x - x^2.$$

This is neither $h(x) = 2x - x^2$ nor $-h(x) = -2x + x^2$. Therefore, h is neither even nor odd.

(d) Finally,

$$k(-x) = |-x| = |x| = k(x).$$

Thus, k is even.

Problem 14. Consider the functions $x \mapsto x^2$ and $x \mapsto x^3$.

(a) Does $f : \mathbb{R} \to [0, +\infty)$, $f(x) = x^2$, have an inverse?

(b) Restrict the domain of f so that it has an inverse, and find that inverse.

(c) Show that $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$, has an inverse and find it.

Solution. **(a)** The function f is not one-to-one on $\mathbb{R}$, since

$$f(1) = f(-1) = 1.$$

Therefore, it does not have an inverse on $\mathbb{R}$.

(b) Consider instead

$$f : [0, +\infty) \longrightarrow [0, +\infty), \qquad f(x) = x^2.$$

This restriction is one-to-one and onto. Solving $y = x^2$ with $x \geqslant 0$ gives $x = \sqrt{y}$. Hence,

$$f^{-1}(x) = \sqrt{x}, \qquad x \geqslant 0.$$

Indeed,

$$f^{-1}(f(x)) = \sqrt{x^2} = x \quad (x \geqslant 0), \qquad f(f^{-1}(x)) = (\sqrt{x})^2 = x.$$

(c) The function $g(x) = x^3$ is strictly increasing on $\mathbb{R}$, so it is one-to-one. Moreover, for every $y \in \mathbb{R}$, the number $x = \sqrt[3]{y}$ satisfies $x^3 = y$, so g is onto. Therefore,

$$g^{-1}(x) = \sqrt[3]{x}.$$

Notice that f^{-1} denotes the inverse function; it does not denote the reciprocal $1/f$.

Problem 15. Sketch the graphs of

$$f(x) = \left|x^2 - 2x - 3\right|, \qquad g(x) = |x-1| + |x+1|.$$

Solution. For the first function, put

$$h(x) = x^2 - 2x - 3 = (x+1)(x-3).$$

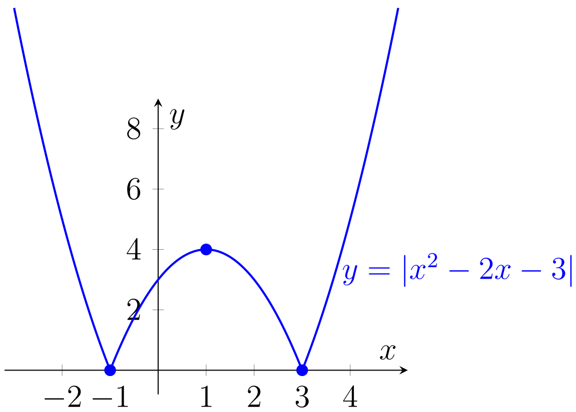
The zeros of h are -1 and 3. Moreover, $h(x) \geqslant 0$ outside the interval $[-1, 3]$ and $h(x) < 0$ inside it. Therefore,

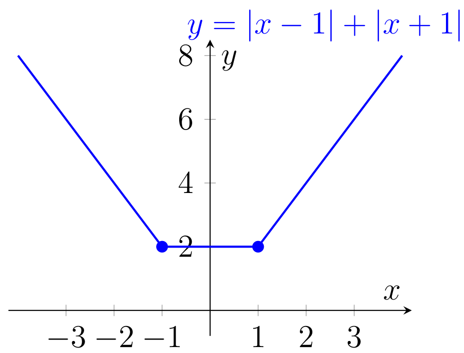
$$f(x) = \begin{cases} x^2 - 2x - 3, & x \leqslant -1, \\ -x^2 + 2x + 3, & -1 < x < 3, \\ x^2 - 2x - 3, & x \geqslant 3. \end{cases}$$

The part of the parabola $y = x^2 - 2x - 3$ below the x-axis is reflected above the x-axis. The graph has zeros at $x = -1$ and $x = 3$, and the reflected central arc has its maximum at $(1, 4)$.

For g, the expressions inside the absolute values change sign at -1 and 1. Thus,

$$g(x) = \begin{cases} -2x, & x < -1, \\ 2, & -1 \leqslant x < 1, \\ 2x, & x \geqslant 1. \end{cases}$$





Problem 16. Decide whether each of the following functions is even, odd or neither:

(a) $f(x) = x^2 + x^3$.

(b) $g(x) = \ln\left(\dfrac{1-x}{1+x}\right)$.

(c) $h(x) = \dfrac{\sin x}{x}$.

Solution. **(a)** The domain of f is $\mathbb{R}$, which is symmetric. We have

$$f(-x) = x^2 - x^3.$$

This is neither $f(x) = x^2 + x^3$ nor $-f(x) = -x^2 - x^3$. Hence, f is neither even nor odd.

(b) We first determine the domain. The logarithm requires

$$\frac{1-x}{1+x} > 0.$$

The critical points are -1 and 1, and the quotient is positive precisely when $-1 < x < 1$. Thus,

$$\mathrm{Dom}(g) = (-1, 1),$$

which is symmetric. For every $x \in (-1, 1)$,

$$\begin{aligned} g(-x) &= \ln\left(\frac{1+x}{1-x}\right) = \ln\left(\left(\frac{1-x}{1+x}\right)^{-1}\right) \\ &= -\ln\left(\frac{1-x}{1+x}\right) = -g(x). \end{aligned}$$

Therefore, g is odd.

(c) The domain is $\mathrm{Dom}(h) = \mathbb{R} \setminus \{0\}$, which is symmetric. Since both $\sin x$ and x are odd,

$$h(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = h(x).$$

Hence, h is even.

Problem 17. Find the natural domain of each function:

(a) $f(x) = \sqrt{\dfrac{2x-1}{x+1} - 1}$;

(b) $g(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$;

(c) $h(x) = \sqrt{1 - |x|}$;

(d) $k(x) = e^{\sqrt{\frac{x-1}{2-x}}}$;

(e) $p(x) = \sqrt[x-3]{x^2 - 1}$;

(f) $q(x) = \ln(\ln x) + \sqrt{\dfrac{x^2-4}{x-3}}$.

Solution. **(a)** We require $x \neq -1$ and

$$\frac{2x-1}{x+1} - 1 = \frac{x-2}{x+1} \geqslant 0.$$

The critical points are -1 and 2. The quotient is positive on $(-\infty, -1)$ and $(2, +\infty)$, and it vanishes at 2. Therefore,

$$\mathrm{Dom}(f) = (-\infty, -1) \cup [2, +\infty).$$

(b) Since the square root appears in the denominator, its radicand must be strictly positive:

$$x^2 - 3x + 2 = (x-1)(x-2) > 0.$$

This product is positive outside the interval between its roots. Hence,

$$\mathrm{Dom}(g) = (-\infty, 1) \cup (2, +\infty).$$

(c) The condition on the radicand is

$$1 - |x| \geqslant 0 \quad \Longleftrightarrow \quad |x| \leqslant 1.$$

Consequently, $\mathrm{Dom}(h) = [-1, 1]$.

(d) The exponential introduces no additional restriction. We only need

$$\frac{x-1}{2-x} \geqslant 0, \qquad x \neq 2.$$

The critical points are 1 and 2. The quotient is non-negative precisely for $1 \leqslant x < 2$, and thus $\mathrm{Dom}(k) = [1, 2)$.

(e) Following the definition of a general real power, we write

$$p(x) = (x^2 - 1)^{1/(x-3)} = e^{\frac{\ln(x^2-1)}{x-3}}.$$

We require $x^2 - 1 > 0$ and $x \neq 3$. The first condition is equivalent to $x < -1$ or $x > 1$. Therefore,

$$\mathrm{Dom}(p) = (-\infty, -1) \cup (1, 3) \cup (3, +\infty).$$

(f) The expression $\ln(\ln x)$ requires $\ln x > 0$, which is equivalent to $x > 1$. For the square root, we need

$$\frac{x^2-4}{x-3} = \frac{(x-2)(x+2)}{x-3} \geqslant 0, \qquad x \neq 3.$$

The critical points are -2, 2 and 3. A sign chart gives

$$\frac{x^2-4}{x-3} \geqslant 0 \quad \Longleftrightarrow \quad x \in [-2, 2] \cup (3, +\infty).$$

Intersecting this set with $(1, +\infty)$ gives $\mathrm{Dom}(q) = (1, 2] \cup (3, +\infty)$.