Calculus 1

Lecture 5: The Derivative

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- ⋆ Explain why differentiability implies continuity.
- ⋆ Recognize corners and endpoint derivatives.
- ⋆ Differentiate powers and basic elementary functions.

From average change to the derivative

A secant slope measures average change

The points

$$(a, f(a)) \qquad \text{and} \qquad (a+h, f(a+h))$$

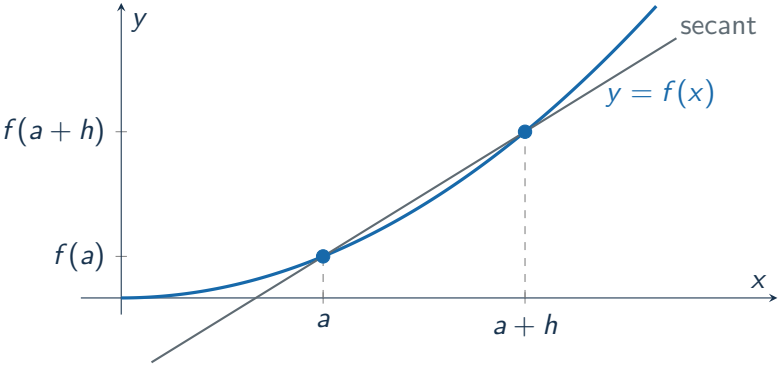
determine a secant line.

Its slope is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

This quotient measures the average output change per unit input change.

The second point moves towards the first



As $h \to 0$, the second point approaches $(a, f(a))$.

A finite limiting slope defines the derivative

Let $f : A \to \mathbb{R}$, and let $a \in A$ be approachable by other points of A.

The function is **differentiable at** a if the finite limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, with $a + h \in A$.

The number $f'(a)$ is the **derivative of** f **at** a.

The definition has two equivalent forms

Putting $x = a + h$ gives $h = x - a$. Therefore,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The h-form

Useful for calculations and rates of change.

The x-form

Makes the two points of the secant line visible.

The derivative may itself be a function

If the derivative exists at every point of a set, we write

$$f' : x \longmapsto f'(x).$$

The domain of f' may be smaller than the domain of f.

For example, $f(x) = |x|$ is defined and continuous on $\mathbb{R}$, but its derivative does not exist at $x = 0$.

Geometric and physical meaning

The derivative is the slope of the tangent line

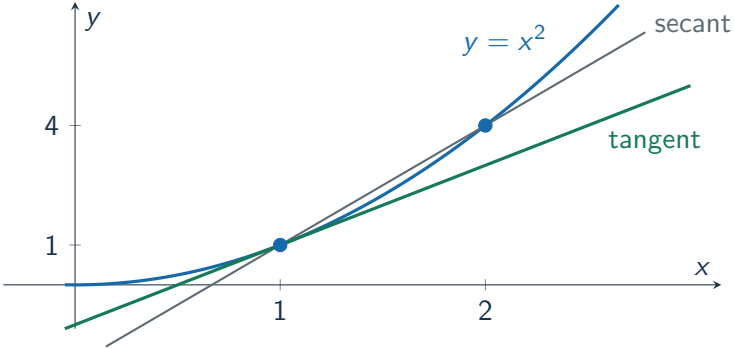
If $f'(a)$ exists, the tangent line to $y = f(x)$ at $(a, f(a))$ has slope $f'(a)$.

Using the point–slope equation, its equation is

$$\boxed{y = f(a) + f'(a)(x - a)}.$$

The limit of secant slopes determines the tangent slope.

Secant slopes approach the tangent slope



One intersection does not define tangency

Consider the curve $y = x^3$.

At the origin, the tangent line is $y = 0$. The curve crosses this tangent line.

The line $y = 1$ meets the curve only at $(1, 1)$, but it is not tangent there:

$$\text{slope of } y = 1 = 0, \qquad \left.\frac{d}{dx}x^3\right|_{x=1} = 3.$$

Tangency comes from a limiting slope, not from counting intersections.

Position gives velocity through the same limit

If $s(t)$ is the position of a particle, then

$$\frac{s(t+h)-s(t)}{h}$$

is its average velocity from time t to time $t+h$.

If the limit exists, then

$$v(t) = s'(t)$$

is its instantaneous velocity.

If v is differentiable, then $a(t) = v'(t) = s''(t)$ is the acceleration.

Example: find an instantaneous velocity

Let

$$s(t) = t^3 - 6t^2 + 9t$$

be the position in metres.

Find the average velocity from $t = 1$ to $t = 1 + h$, and then find the instantaneous velocity at $t = 1$.

First calculate the two position values

At $t = 1$,

$$s(1) = 1 - 6 + 9 = 4.$$

At $t = 1 + h$,

$$\begin{aligned} s(1+h) &= (1+h)^3 - 6(1+h)^2 + 9(1+h) \\ &= 4 - 3h^2 + h^3. \end{aligned}$$

Then simplify the average velocity

$$
\begin{aligned}
\frac{s(1+h)-s(1)}{h} &= \frac{4-3h^2+h^3-4}{h} \\
&= \frac{-3h^2+h^3}{h} \\
&= -3h+h^2, \qquad h \neq 0.
\end{aligned}
$$

The limit gives the instantaneous velocity

Now let the time interval shrink:

$$s'(1) = \lim_{h \to 0} (-3h + h^2) = 0.$$

At $t = 1$, the particle is instantaneously at rest.

Its instantaneous velocity is 0 metres per second.

Derivatives from first principles

A first-principles calculation follows four steps

To calculate $f'(a)$ from the definition:

1. Form $f(a+h) - f(a)$.
2. Divide by h, keeping $h \neq 0$.
3. Simplify before taking the limit.
4. Let $h \to 0$ and record the domain.

Direct substitution at the beginning usually produces $0/0$.

Example: differentiate $f(x) = x^2$

At an arbitrary point a,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}.$$

The variable a stays fixed while h approaches zero.

Expand before taking the limit

$$
\begin{aligned}
\frac{(a+h)^2 - a^2}{h} &= \frac{a^2 + 2ah + h^2 - a^2}{h} \\
&= \frac{2ah + h^2}{h} \\
&= 2a + h, \qquad h \neq 0.
\end{aligned}
$$

Now the limit is immediate

After simplifying the quotient,

$$f'(a) = \lim_{h \to 0} (2a + h) = 2a.$$

Since a was arbitrary,

$$\boxed{f'(x) = 2x \qquad \text{for every } x \in \mathbb{R}.}$$

At $a = 1$, the derivative gives the tangent line

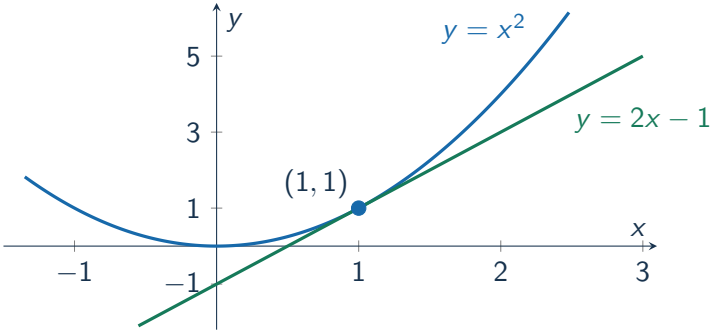
For $f(x) = x^2$,

$$f(1) = 1, \qquad f'(1) = 2.$$

The tangent line is therefore

$$\begin{aligned} y &= f(1) + f'(1)(x-1) \\ &= 1 + 2(x-1) \\ &= 2x - 1. \end{aligned}$$

The graph and the tangent share position and slope



Example: differentiate a reciprocal

Let

$$f(x) = \frac{1}{x}$$

and fix $a \neq 0$.

For sufficiently small h, we also have $a + h \neq 0$. Thus,

$$f'(a) = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}.$$

Combine the fractions before taking the limit

$$
\begin{aligned}
\frac{\frac{1}{a+h} - \frac{1}{a}}{h} &= \frac{\frac{a-(a+h)}{a(a+h)}}{h} \\
&= \frac{-h}{a(a+h)h} \\
&= -\frac{1}{a(a+h)}, \qquad h \neq 0.
\end{aligned}
$$

The original domain remains part of the answer

We obtain

$$f'(a) = \lim_{h \to 0} -\frac{1}{a(a+h)} = -\frac{1}{a^2}.$$

Therefore,

$$\boxed{\left(\frac{1}{x}\right)' = -\frac{1}{x^2}, \qquad x \neq 0.}$$

The derivative formula does not add $x = 0$ to the original domain.

Example: begin with a natural power

Let $f(x) = x^n$, where $n \in \mathbb{N}$. From the definition,

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^n - a^n}{h}.$$

The binomial theorem will reveal a factor of h in the numerator.

The binomial expansion isolates the linear term

The binomial theorem gives

$$(a+h)^n = a^n + na^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + h^n.$$

Hence,

$$\frac{(a+h)^n - a^n}{h} = na^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + h^{n-1}.$$

Every remaining power of h vanishes

Letting $h \to 0$ gives

$$f'(a) = na^{n-1}.$$

Therefore,

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}, \qquad n \in \mathbb{N}.}$$

This argument proves the power rule for natural exponents. Other exponents require additional arguments and domain care.

Differentiability and continuity

Differentiability is stronger than continuity

If f is differentiable at a, then f is continuous at a.

The derivative controls the quotient

$$\frac{f(x) - f(a)}{x - a}.$$

Multiplying it by the small quantity $x - a$ controls the difference $f(x) - f(a)$.

Factor the change in the function

For $x \neq a$,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a).$$

As $x \to a$,

$$\frac{f(x) - f(a)}{x - a} \longrightarrow f'(a)$$

because f is differentiable at a.

Take the limit of the product

We also have

$$x - a \longrightarrow 0.$$

Consequently,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a) \longrightarrow f'(a) \cdot 0 = 0.$$

Thus,

$$\lim_{x \to a} f(x) = f(a),$$

so f is continuous at a.

The converse is false

Continuity does not guarantee differentiability.

A continuous graph may have

- a corner, as for $f(x) = |x|$ at 0;
- a cusp, as for $f(x) = |x|^{2/3}$ at 0;
- a vertical tangent, as for $f(x) = x^{1/3}$ at 0.

In each case, continuity survives but a finite two-sided derivative does not.

Example: test differentiability of $|x|$ at the origin

The function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} |x| = 0 = f(0).$$

To test differentiability, examine

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h}.$$

The right derivative equals 1

If $h > 0$, then $|h| = h$. Therefore,

$$\frac{|h|}{h} = 1.$$

Hence,

$$f'_+(0) = \lim_{h \to 0^+} \frac{|h|}{h} = 1.$$

The left derivative equals -1

If $h < 0$, then $|h| = -h$. Therefore,

$$\frac{|h|}{h} = -1.$$

Hence,

$$f'_-(0) = \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$

Different one-sided derivatives create a corner

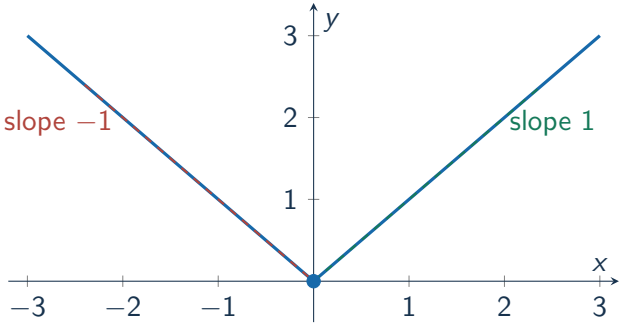
Since

$$f'_-(0) = -1 \qquad \text{and} \qquad f'_+(0) = 1,$$

the derivative $f'(0)$ does not exist.
Away from the origin,

$$\frac{d}{dx}|x| = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

The graph has two different tangent directions



Endpoint derivatives use the available side

For a function defined on $[a, b]$, we may define

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

and

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}.$$

At an interior point, differentiability requires both one-sided derivatives to exist and to be equal.

Linearity and the power rule

Basic rules follow from the algebra of limits

If f and g are differentiable and $c \in \mathbb{R}$, then

$$\begin{aligned}(c)' &= 0,\\ (cf)' &= cf',\\ (f+g)' &= f' + g',\\ (f-g)' &= f' - g'.\end{aligned}$$

These rules apply to sums and constant multiples. Products and quotients need their own rules, which come next.

The power rule must respect the domain

Wherever the real power is defined and differentiable,

$$\frac{d}{dx}x^{\alpha} = \alpha x^{\alpha-1}.$$

For example,

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}, \qquad x > 0,$$

although $\sqrt{x}$ itself is defined at $x = 0$.
Also,

$$(x^{-1})' = -x^{-2}, \qquad x \neq 0.$$

Example: differentiate a polynomial

Let

$$p(x) = 4x^5 - 3x^2 + 7x - 9.$$

The expression is a sum of monomials, so linearity and the power rule apply term by term. No product rule is needed.

Differentiate one term at a time

$$\begin{aligned} p'(x) &= (4x^5)' - (3x^2)' + (7x)' - (9)' \\ &= 4 \cdot 5x^4 - 3 \cdot 2x + 7 - 0 \\ &= 20x^4 - 6x + 7. \end{aligned}$$

$$p'(x) = 20x^4 - 6x + 7 \text{ for every } x \in \mathbb{R}.$$

Derivatives of elementary functions

Algebraic, exponential and logarithmic derivatives

$f(x)$	$f'(x)$	domain information
x^α	$\alpha x^{\alpha-1}$	where the power is differentiable
e^x	e^x	$x \in \mathbb{R}$
a^x	$a^x \ln a$	$a > 0$
$\ln x$	$1/x$	$x > 0$

The derivative formula never removes a restriction from the original function.

The basic trigonometric derivatives

$f(x)$	$f'(x)$	domain information
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$1/\cos^2 x$	$\cos x \neq 0$

Angles must be measured in radians for these formulas to hold in this form.

Why is the derivative of sine equal to cosine?

From the definition and the addition formula,

$$\begin{aligned}\frac{\sin(a+h)-\sin a}{h} &= \frac{\sin a\cos h+\cos a\sin h-\sin a}{h}\\ &= \sin a\frac{\cos h-1}{h}+\cos a\frac{\sin h}{h}.\end{aligned}$$

The standard trigonometric limits finish the proof

Recall that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0, \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Therefore,

$$\begin{aligned}(\sin x)'\big|_{x=a} &= \sin a \cdot 0 + \cos a \cdot 1 \\ &= \cos a.\end{aligned}$$

Hence,

$$\boxed{(\sin x)' = \cos x.}$$

Example: combine linearity with the table

Let

$$f(x) = 3e^x - 2\sin x + \ln x.$$

Before differentiating, record the natural domain:

$$\mathrm{Dom}(f) = (0, +\infty),$$

because of the logarithm.

Differentiate term by term

$$
\begin{aligned}
f'(x) &= (3e^x)' - (2\sin x)' + (\ln x)' \\
&= 3e^x - 2\cos x + \frac{1}{x}.
\end{aligned}
$$

The original domain remains in force:

$$
\boxed{f'(x) = 3e^x - 2\cos x + \frac{1}{x}, \qquad x > 0.}
$$

A reliable differentiation checklist

Before differentiating a formula:

1. Record its natural domain.
2. Identify sums, products, quotients and compositions.
3. Use each rule only where it applies.
4. Simplify without enlarging the original domain.
5. Check piecewise functions carefully.

Closing summary

The derivative describes local linear change

- The derivative is the limit of a difference quotient.
- Geometrically, it is a tangent slope.
- Physically, it is an instantaneous rate with meaningful units.
- Differentiability implies continuity, but continuity is not enough.
- Linearity, the power rule and a short table handle basic formulas.

Final check

1. Write both equivalent definitions of $f'(a)$.
2. Find the tangent line to $y = x^2$ at $x = 1$.
3. Explain why $|x|$ is continuous but not differentiable at 0.
4. State the derivative and domain of $1/x$.
5. Differentiate $3e^x - 2\sin x + \ln x$ on its natural domain.