Calculus 1

Lecture 6: Differentiation Rules and Related Rates

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- ⋆ Apply logarithmic differentiation to variable powers.
- ⋆ Find tangent slopes from implicit equations.

The product and quotient rules

The product of the derivatives is usually wrong

Suppose that

$$f(x) = x, \qquad g(x) = x.$$

Then

$$(fg)(x) = x^2 \quad \Longrightarrow \quad (fg)'(x) = 2x.$$

But

$$f'(x)g'(x) = 1 \cdot 1 = 1.$$

Thus, in general,

$$(fg)' \neq f'g'.$$

The product rule contains two terms

If f and g are differentiable at x, then

$$\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x).}$$

Both factors may change. The formula records the contribution from changing f and the contribution from changing g.

Why do two terms appear? - For the oral exam

Start with the difference quotient

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Add and subtract the intermediate term $f(x+h)g(x)$:

$$\begin{aligned} &f(x+h)g(x+h) - f(x)g(x) \\ &\quad = f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x)). \end{aligned}$$

Splitting the quotient gives the product rule

After dividing by h,

$$\frac{f(x+h)g(x+h)-f(x)g(x)}{h} = f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}.$$

Letting $h \to 0$ and using continuity of f gives

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

A verbal pattern helps organize the calculation

The product rule may be remembered as

> differentiate the first, keep the second + keep the first, differentiate the second.

The order of the two terms does not matter. Missing one of them does.

Example: differentiate $x^2 e^x$

Let

$$F(x) = x^2 e^x.$$

Choose the two factors

$$f(x) = x^2, \qquad g(x) = e^x.$$

Then

$$f'(x) = 2x, \qquad g'(x) = e^x.$$

Apply the product rule before simplifying

$$F'(x) = (x^2)'e^x + x^2(e^x)'$$
$$= 2xe^x + x^2e^x.$$

Both terms contain e^x, so

$$\boxed{F'(x) = e^x(x^2 + 2x).}$$

The quotient rule also records two changes

If f and g are differentiable at x and $g(x) \neq 0$, then

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.}$$

The order in the numerator matters:

derivative of the numerator $\times$ denominator $-$ numerator $\times$ derivative of the denominator.

Example: differentiate a trigonometric quotient

Let

$$G(x) = \frac{\sin x}{1 + x^2}.$$

The denominator satisfies

$$1 + x^2 > 0$$

for every $x \in \mathbb{R}$. Thus, G is defined on all of $\mathbb{R}$.

Set

$$f(x) = \sin x, \qquad g(x) = 1 + x^2.$$

Insert each derivative into the quotient rule

We have

$$f'(x) = \cos x, \qquad g'(x) = 2x.$$

Therefore,

$$\begin{aligned} G'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos x(1+x^2) - \sin x(2x)}{(1+x^2)^2}. \end{aligned}$$

The differentiated quotient has the same domain

Reordering the factors gives

$$\boxed{G'(x) = \frac{(1+x^2)\cos x - 2x \sin x}{(1+x^2)^2}.}$$

Since $1 + x^2$ never vanishes, this derivative exists for every $x \in \mathbb{R}$.

The chain rule

A composition applies functions in layers

Consider

$$h(x) = \sqrt{1 + x^2}.$$

To evaluate it, we

1. square x;
2. add 1;
3. take the square root.

The chain rule differentiates these layers from the outside inward.

The chain rule differentiates a composition

Suppose that f is differentiable at a and that g is differentiable at $f(a)$. Then

$$\boxed{(g \circ f)'(a) = g'(f(a))f'(a).}$$

In function notation,

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x).$$

Leibniz notation displays the intermediate variable

Write

$$y = g(u), \qquad u = f(x).$$

Then the chain rule takes the form

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.}$$

This notation is useful for organizing the factors. It represents the chain rule, not an ordinary cancellation of fractions.

Example: identify the inner and outer functions

For

$$h(x) = \sqrt{1 + x^2} = (1 + x^2)^{1/2},$$

choose

$$u(x) = 1 + x^2$$

as the inner function and

$$g(u) = u^{1/2}$$

as the outer function.

Differentiate the outside and multiply by the inside

We have

$$g'(u) = \frac{1}{2}u^{-1/2}, \qquad u'(x) = 2x.$$

Therefore,

$$\begin{aligned} h'(x) &= \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x \\ &= \frac{x}{\sqrt{1+x^2}}. \end{aligned}$$

Example: a function with three layers

Consider

$$F(x) = \cos(e^{x^2}).$$

The layers are

$$x \longmapsto x^2 \longmapsto e^{x^2} \longmapsto \cos(e^{x^2}).$$

We will write one derivative factor for each layer.

Differentiate from the outside inward

The outermost function is cosine:

$$\frac{d}{dx}\cos(e^{x^2}) = -\sin(e^{x^2}) \cdot \frac{d}{dx}e^{x^2}.$$

The exponential still contains the inner function x^2:

$$\frac{d}{dx}e^{x^2} = e^{x^2} \cdot 2x.$$

One factor appears for each layer

Combining the factors,

$$\boxed{F'(x) = -\sin(e^{x^2})\, e^{x^2}\, 2x.}$$

An equivalent simplified form is

$$F'(x) = -2xe^{x^2} \sin(e^{x^2}).$$

Useful exponential and logarithmic chain rules

For a differentiable function u,

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} u'(x).$$

Where $u(x) > 0$,

$$\frac{d}{dx} \ln(u(x)) = \frac{u'(x)}{u(x)}.$$

The condition $u(x) > 0$ belongs to the logarithmic formula.

Useful trigonometric and power chain rules

Where the expressions are defined and differentiable,

$$\frac{d}{dx}\sin(u(x)) = \cos(u(x))u'(x),$$

and

$$\frac{d}{dx}(u(x))^{\alpha} = \alpha(u(x))^{\alpha-1}u'(x).$$

The final factor $u'(x)$ records the change in the inner function.

Example: choose the rules before calculating

Differentiate

$$H(x) = x^2 \sin(x^3).$$

The outermost operation is a product:

$$\underbrace{x^2}_{\text{first factor}} \cdot \underbrace{\sin(x^3)}_{\text{second factor}} .$$

The second factor also requires the chain rule.

Use the product rule and then the chain rule

$$\begin{aligned} H'(x) &= (x^2)' \sin(x^3) + x^2 (\sin(x^3))' \\ &= 2x \sin(x^3) + x^2 \cos(x^3) \cdot 3x^2 \\ &= 2x \sin(x^3) + 3x^4 \cos(x^3). \end{aligned}$$

Derivatives of inverse functions

An inverse function satisfies an identity

If f is one-to-one, then

$$f^{-1}(f(x)) = x.$$

Differentiating both sides with the chain rule suggests

$$(f^{-1})'(f(x))f'(x) = 1.$$

Therefore,

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)},$$

provided that $f'(x) \neq 0$ and the inverse satisfies the required hypotheses.

The logarithm inherits its derivative from the exponential

(This can be for the oral exam!) Let

$$f(x) = e^x, \qquad f^{-1}(x) = \ln x.$$

Since $f'(x) = e^x$, the inverse rule gives

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}, \qquad x > 0.$$

Example: an inverse derivative without an inverse formula

Let

$$f(x) = x + e^x.$$

Since

$$f'(x) = 1 + e^x > 0,$$

the function is strictly increasing and therefore one-to-one.
Also,

$$f(0) = 1.$$

Evaluate the original derivative at the matching point

We have

$$f'(0) = 1 + e^0 = 2.$$

The inverse rule now gives

$$(f^{-1})'(f(0)) = \frac{1}{f'(0)}.$$

Therefore,

$$\boxed{(f^{-1})'(1) = \frac{1}{2}.}$$

Implicit differentiation

An equation may define a curve without one global formula

The equation

$$x^2 + y^2 = 1$$

describes a circle.

It cannot be represented by one function $y = f(x)$ on the whole circle. Locally, however, each nonvertical branch behaves like a function.

We treat y as a function $y(x)$ and differentiate both sides.

Differentiate the circle equation

Starting from

$$x^2 + y^2 = 1,$$

differentiate with respect to x:

$$2x + 2y\frac{dy}{dx} = 0.$$

The chain rule produces the factor dy/dx when differentiating y^2.

Solve for the tangent slope

From

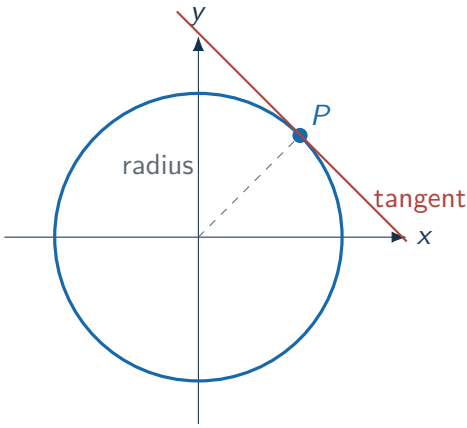
$$2x + 2yy' = 0,$$

we obtain

$$\boxed{y' = -\frac{x}{y}, \qquad y \neq 0.}$$

At $(1, 0)$ and $(-1, 0)$, this formula fails because the tangent is vertical.

The circle tangent is perpendicular to the radius



Example: find a tangent to an implicit curve

Find the tangent line to

$$x^2 + xy + y^2 = 7$$

at $(1, 2)$.

First verify the point:

$$1^2 + 1 \cdot 2 + 2^2 = 1 + 2 + 4 = 7.$$

Differentiate every term with respect to x

Starting from

$$x^2 + xy + y^2 = 7,$$

we obtain

$$2x + (xy)' + 2yy' = 0.$$

The term xy requires the product rule:

$$(xy)' = 1 \cdot y + x \cdot y' = y + xy'.$$

Therefore,

$$2x + y + xy' + 2yy' = 0.$$

Collect all terms containing y'

From

$$2x + y + xy' + 2yy' = 0,$$

we get

$$(x + 2y)y' = -(2x + y).$$

Thus,

$$\boxed{y' = -\frac{2x + y}{x + 2y}.}$$

Evaluate the slope and write the tangent line

At $(1, 2)$,

$$y' = -\frac{2(1) + 2}{1 + 2(2)} = -\frac{4}{5}.$$

Using point–slope form,

$$\boxed{y - 2 = -\frac{4}{5}(x - 1).}$$

The tangent slope belongs to the local branch passing through $(1, 2)$.

Closing summary

The outermost operation determines the first rule

Algebraic structure

sum $\longrightarrow$ linearity

product $\longrightarrow$ product rule

quotient $\longrightarrow$ quotient rule

Other structures

composition $\longrightarrow$ chain rule

relation in x and y $\longrightarrow$ implicit differentiation

variable power $\longrightarrow$ logarithmic differentiation

Read the structure first; then choose the rule.

The rules work together

- ⋆ Product and quotient rules track two changing expressions.
- ⋆ The chain rule follows nested functions from outside to inside.
- ⋆ The inverse rule converts a known slope into its reciprocal.
- ⋆ Logarithms simplify variable powers.
- ⋆ Implicit differentiation handles curves.

Final check

1. Differentiate $x^2 e^x$.
2. Differentiate $\cos(e^{x^2})$ and identify its three layers.
3. Find $(f^{-1})'(1)$ when $f(x) = x + e^x$.
4. Differentiate x^x for $x > 0$.
5. Find the tangent slope to $x^2 + xy + y^2 = 7$ at $(1, 2)$.