

7 Lecture #5: Week 3

7.1 From a rate of change to the derivative

In the previous lecture we considered the slope of a secant line through $(a, f(a))$ and $(a + h, f(a + h))$. That slope is

$$\frac{f(a + h) - f(a)}{h}.$$

It measures the average change in the output per unit change in the input. If the quotient approaches a finite value as $h \rightarrow 0$, we obtain an instantaneous rate of change. This is the central object of differential calculus.

Definition 7.1. Let $f : A \rightarrow \mathbb{R}$, and let $a \in A$ be a point that can be approached by other points of A . The function f is **differentiable at** a if the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

exists, with $a + h \in A$. The number $f'(a)$ is the **derivative of f at a** . The function that assigns $x \mapsto f'(x)$ is denoted by f' .

By putting $x = a + h$, the same definition can be written as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Both forms are important. The h -form is convenient for calculations and physical interpretations, while the x -form makes the secant line visible.

7.2 Geometric and physical meaning

If $f'(a)$ exists, it is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$. Consequently, the tangent line has equation

$$y = f(a) + f'(a)(x - a).$$

This is a definition through a limiting process. It is not correct to define a tangent as a line that meets the curve at only one point: a tangent may cross the curve, and a non-tangent line may happen to meet it only once.

If $s(t)$ is the position of a particle, then

$$\frac{s(t + h) - s(t)}{h}$$

is its average velocity over the time interval from t to $t + h$, and $s'(t)$ is its instantaneous velocity. If $v(t) = s'(t)$ is differentiable, then $v'(t) = s''(t)$ is the acceleration. The same idea applies to any pair of changing quantities, and the units of the derivative are always “units of output per unit of input.”

Example 7.2. Let $s(t) = t^3 - 6t^2 + 9t$ metres. The average velocity from $t = 1$ to $t = 1 + h$ is

$$\begin{aligned}\frac{s(1+h) - s(1)}{h} &= \frac{(1+h)^3 - 6(1+h)^2 + 9(1+h) - 4}{h} \\ &= \frac{-3h^2 + h^3}{h} = -3h + h^2.\end{aligned}$$

Therefore, the instantaneous velocity at $t = 1$ is 0 metres per second. The particle is instantaneously at rest at that time.

7.3 Calculating derivatives from first principles

The definition, rather than a table, tells us what a derivative actually is. Let us use it before introducing shortcuts.

Example 7.3 (Derivative of a quadratic function). Let $f(x) = x^2$. At an arbitrary point a ,

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2ah + h^2}{h} = \lim_{h \rightarrow 0} (2a + h) = 2a.\end{aligned}$$

Since a was arbitrary, $f'(x) = 2x$ for every $x \in \mathbb{R}$.

At $a = 1$, the tangent slope is 2 and the tangent line is

$$y = 1 + 2(x - 1) = 2x - 1.$$

Example 7.4 (Derivative of a reciprocal). Let $f(x) = 1/x$ and fix $a \neq 0$. For sufficiently small h , also $a + h \neq 0$, and

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{1/(a+h) - 1/a}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{a(a+h)h} = \lim_{h \rightarrow 0} -\frac{1}{a(a+h)} = -\frac{1}{a^2}.\end{aligned}$$

Thus,

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}, \quad x \neq 0.$$

The domain is part of the answer.

Example 7.5 (Derivative of a natural power). For $f(x) = x^n$, the binomial theorem gives

$$(a+h)^n - a^n = na^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + h^n.$$

After dividing by h and letting $h \rightarrow 0$, every term except the first vanishes. Hence,

$$\frac{d}{dx}x^n = nx^{n-1}, \quad n \in \mathbb{N}.$$

The same power rule is valid for every real exponent on intervals where the power is defined and differentiable, but the argument above proves only the natural case.

7.4 Differentiability implies continuity

The derivative is a stronger local property than continuity.

Theorem 7.6. If f is differentiable at a , then f is continuous at a .

Proof. For $x \neq a$, write

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a).$$

As $x \rightarrow a$, the first factor tends to $f'(a)$ and the second tends to zero. Therefore, $f(x) - f(a) \rightarrow 0$, which means $f(x) \rightarrow f(a)$. $\square$

The converse is false. A function may be continuous and still have a corner, a cusp, or a vertical tangent.

Example 7.7. The absolute-value function is continuous at 0, but

$$\lim_{h \rightarrow 0^+} \frac{|h| - |0|}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|h| - |0|}{h} = -1.$$

The one-sided derivatives do not agree, so $|x|$ is not differentiable at 0. Away from 0 it is locally either x or $-x$, and hence

$$\frac{d}{dx}|x| = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

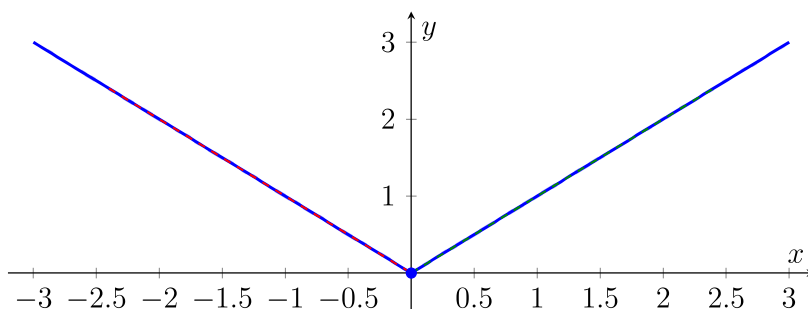


Figure 26: The graph of $|x|$ has different left and right tangent slopes at the origin.

For a function defined on $[a, b]$, it is common to speak of a right derivative at a and a left derivative at b . At an interior point, differentiability requires both one-sided derivatives to exist and to be equal.

7.5 Linearity and the power rule

The first differentiation rules follow directly from the algebra of limits.

Proposition 7.8 (Basic differentiation rules). Let f and g be differentiable and let $c \in \mathbb{R}$. Then

$$(c)' = 0, \quad (cf)' = cf', \quad (f + g)' = f' + g', \quad (f - g)' = f' - g'.$$

Moreover, wherever x^α is defined and differentiable,

$$\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}.$$

Example 7.9. For

$$p(x) = 4x^5 - 3x^2 + 7x - 9,$$

we differentiate term by term:

$$p'(x) = 20x^4 - 6x + 7.$$

No product rule is needed: the expression is a sum of monomials.

7.6 Derivatives of elementary functions

Together with the rules, we need a small table of basic derivatives. The domains in the right-hand column must be respected.

$f(x)$	$f'(x)$	relevant domain
x^α	$\alpha x^{\alpha-1}$	where the real power is defined
e^x	e^x	$\mathbb{R}$
a^x	$a^x \ln a$	$a > 0$
$\ln x$	$1/x$	$x > 0$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$1/\cos^2 x$	$\cos x \neq 0$

Let us justify one entry. Using the addition formula for sine,

$$\frac{\sin(a+h) - \sin a}{h} = \sin a \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h}.$$

The standard limits from Lecture 3 give zero for the first quotient and one for the second. Therefore, $(\sin x)' = \cos x$.

Example 7.10. Let

$$f(x) = 3e^x - 2\sin x + \ln x, \quad x > 0.$$

By linearity and the table,

$$f'(x) = 3e^x - 2\cos x + \frac{1}{x}.$$

Algorithm 7.11 (First check when differentiating). Before differentiating a formula, do the following.

- (1) Record its natural domain.
- (2) Identify the outermost operations: sum, product, quotient or composition.
- (3) Use linearity and the power rule only where they actually apply.
- (4) Simplify the final expression, but do not enlarge the original domain by accident.
- (5) For a piecewise function, study the junction points separately from the open pieces.