

8 Lecture #6: Week 3

8.1 The product and quotient rules

The derivative of a product is generally *not* the product of the derivatives. Indeed, $(x^2)' = 2x$, while $(x)'(x)' = 1$. The correct formula contains two terms.

Theorem 8.1 (Product and quotient rules). Let f and g be differentiable at x . Then

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

If $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

The product rule can be remembered as “differentiate the first and keep the second, then keep the first and differentiate the second.”

Example 8.2. Let $f(x) = x^2 e^x$. Then

$$f'(x) = 2xe^x + x^2 e^x = e^x(x^2 + 2x).$$

For

$$g(x) = \frac{\sin x}{1 + x^2},$$

the denominator never vanishes, and

$$g'(x) = \frac{(1 + x^2)\cos x - 2x \sin x}{(1 + x^2)^2}.$$

8.2 The chain rule

Most functions occurring in applications are compositions. If $h(x) = \sqrt{1 + x^2}$, we first square x , then add 1, and finally take a square root. The chain rule follows these layers from the outside inward.

Theorem 8.3 (Chain rule). Suppose that f is differentiable at a and that g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

In Leibniz notation, if $y = g(u)$ and $u = f(x)$, this is

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

Example 8.4. For $h(x) = \sqrt{1 + x^2} = (1 + x^2)^{1/2}$, the outer function is $u^{1/2}$ and the inner function is $u = 1 + x^2$. Hence,

$$h'(x) = \frac{1}{2}(1 + x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}.$$

Example 8.5. The function

$$F(x) = \cos(e^{x^2})$$

has three layers. Differentiating from the outside inward gives

$$F'(x) = -\sin(e^{x^2}) \cdot e^{x^2} \cdot 2x.$$

It is often safer to write one factor for each layer before simplifying.

The chain rule also gives useful general formulas wherever all the expressions are defined. In particular, the logarithmic formula requires $u(x) > 0$.

$$\begin{aligned} \frac{d}{dx} e^{u(x)} &= e^{u(x)} u'(x), & \frac{d}{dx} \ln(u(x)) &= \frac{u'(x)}{u(x)}, \\ \frac{d}{dx} \sin(u(x)) &= \cos(u(x)) u'(x), & \frac{d}{dx} (u(x))^\alpha &= \alpha(u(x))^{\alpha-1} u'(x). \end{aligned}$$

8.3 Derivatives of inverse functions

If f is one-to-one, then $f^{-1}(f(x)) = x$. Differentiating this identity with the chain rule suggests

$$(f^{-1})'(f(x)) f'(x) = 1.$$

Theorem 8.6 (Derivative of an inverse). Suppose that f is one-to-one near a , differentiable at a , $f'(a) \neq 0$, and its inverse is continuous at $f(a)$. Then f^{-1} is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Applying this to the exponential function gives

$$(\ln x)' = \frac{1}{x}, \quad x > 0.$$

Applying it to sine restricted to $[-\pi/2, \pi/2]$ gives

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Similarly,

$$(\arctan x)' = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

Example 8.7. Let $f(x) = x + e^x$. Since f is strictly increasing, it has an inverse. We have $f(0) = 1$ and $f'(0) = 1 + e^0 = 2$. Without finding a formula for the inverse, we obtain

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{2}.$$

8.4 Logarithmic differentiation

The ordinary power rule does not directly apply when both the base and exponent depend on x . Suppose that

$$y = f(x)^{g(x)}, \quad f(x) > 0.$$

Taking logarithms converts the power into a product:

$$\ln y = g(x) \ln f(x).$$

Differentiating and then multiplying by y gives

$$y' = f(x)^{g(x)} \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right).$$

Example 8.8. Let $y = x^x$ for $x > 0$. Taking logarithms gives $\ln y = x \ln x$. Therefore,

$$\frac{y'}{y} = \ln x + 1,$$

and hence

$$y' = x^x (\ln x + 1).$$

Example 8.9. For $y = (1 + x^2)^{\cos x}$, the base is positive on $\mathbb{R}$. We have

$$\ln y = \cos x \ln(1 + x^2).$$

Thus,

$$y' = (1 + x^2)^{\cos x} \left(-\sin x \ln(1 + x^2) + \frac{2x \cos x}{1 + x^2} \right).$$

8.5 Implicit differentiation

An equation involving x and y may describe a curve even when it is inconvenient, or impossible globally, to solve for y . We may still calculate tangent slopes by treating y as a function $y(x)$ and differentiating both sides.

For the circle $x^2 + y^2 = 1$, differentiation gives

$$2x + 2yy' = 0,$$

so

$$y' = -\frac{x}{y}, \quad y \neq 0.$$

At a point (x_0, y_0) with $y_0 \neq 0$, this is the slope of the tangent to the corresponding local branch. At $(1, 0)$ and $(-1, 0)$ the formula fails because the tangent is vertical.

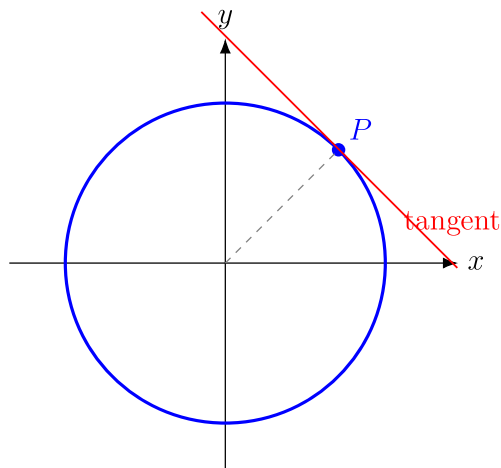


Figure 27: The tangent to $x^2 + y^2 = 1$ is perpendicular to the radius.

Example 8.10. Find the tangent line to

$$x^2 + xy + y^2 = 7$$

at $(1, 2)$. First, $(1, 2)$ belongs to the curve because $1 + 2 + 4 = 7$. Differentiating implicitly,

$$2x + y + xy' + 2yy' = 0.$$

Solving for y' gives

$$y' = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$, the slope is $-4/5$. Hence, the tangent line is

$$y - 2 = -\frac{4}{5}(x - 1).$$

8.6 Related rates

When several quantities depend on time and satisfy an equation, implicit differentiation relates their rates. The essential point is to differentiate with respect to time before inserting the numerical data.

Algorithm 8.11 (Solving a related-rates problem). We follow these steps.

- (1) Draw a picture when geometry is involved.
- (2) Introduce variables for the changing quantities and record their units.
- (3) Write an equation relating the variables.
- (4) Differentiate the equation with respect to time.
- (5) Substitute the data for the instant under consideration and solve for the requested rate.
- (6) Interpret the sign and include units.

Example 8.12 (An expanding circle). The radius of a circular oil spill grows at 3 metres per minute. How fast is its area growing when the radius is 5 metres? Since $A = \pi r^2$,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

At the indicated instant,

$$\frac{dA}{dt} = 2\pi \cdot 5 \cdot 3 = 30\pi \quad \text{square metres per minute.}$$

The area does not grow at a constant rate even though the radius does.

Example 8.13 (A sliding ladder). A ladder 10 metres long rests against a vertical wall. Let $x(t)$ be the distance of its foot from the wall and $y(t)$ the height of its top. Then

$$x^2 + y^2 = 100.$$

Suppose that the foot moves away from the wall at 2 metres per second. When $x = 6$, the Pythagorean theorem gives $y = 8$. Differentiating with respect to time,

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

Therefore,

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{6}{8} \cdot 2 = -\frac{3}{2} \quad \text{metres per second.}$$

The negative sign means that the top of the ladder is moving downward.

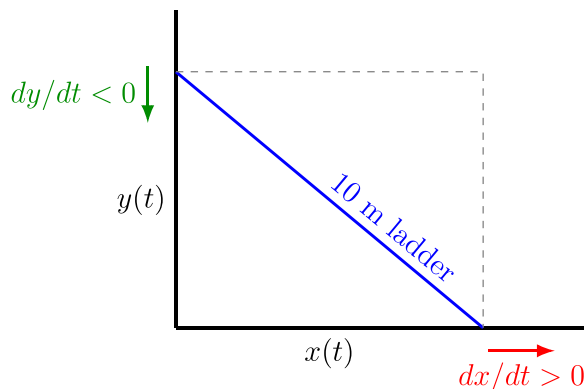


Figure 28: The fixed ladder length relates the horizontal and vertical rates.