

Calculus 1

Practice 3: Derivatives

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Five exercises for the review

- 1 Derivative from first principles
- 2 Reading the derivative from a graph
- 3 Quotient, product, chain and inverse-function rules
- 4 Differentiability of oscillatory functions at the origin
- 5 A tangent line constrained to pass through the origin

Every exercise separates the choice of method from the calculation.

Exercise 1: First principles

Exercise 1

Use the definition of the derivative to find $f'(a)$ when

$$f(x) = x^2.$$

The definition is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

The limit cannot be evaluated by substituting $h = 0$ before simplifying.

Insert the function into the difference quotient

Starting from

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

we obtain

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h}. \end{aligned}$$

The numerator has a factor h , which cancels before the limit is taken.

Simplification reveals the derivative

For $h \neq 0$,

$$\begin{aligned}\frac{(a+h)^2 - a^2}{h} &= \frac{2ah + h^2}{h} \\ &= 2a + h.\end{aligned}$$

Therefore,

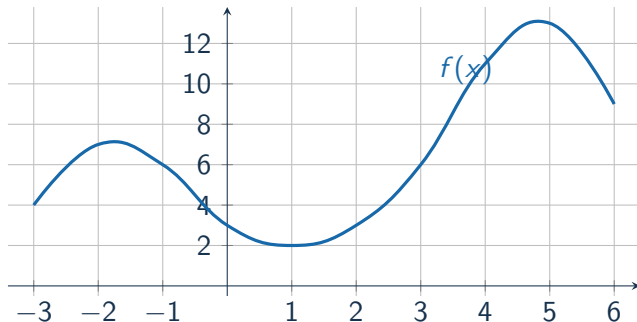
$$f'(a) = \lim_{h \rightarrow 0} (2a + h) = 2a.$$

$$f'(x) = 2x \text{ for every } x \in \mathbb{R}.$$

Exercise 2: Reading a derivative from a graph

Exercise 2

The graph of a differentiable function f is shown below. Sketch a possible graph of f' .



Horizontal tangents locate the zeros of f'

The graph has horizontal tangents at

$$x = -2, \quad x = 1, \quad x = 5.$$

Hence,

$$f'(-2) = f'(1) = f'(5) = 0.$$

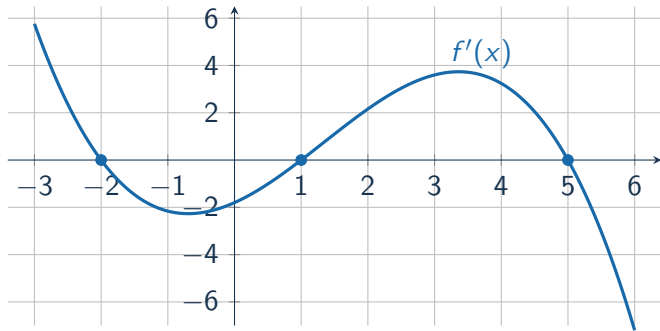
These points divide the real line into intervals on which f is either increasing or decreasing.

Increasing and decreasing determine the sign of f'

Interval	Behaviour of f	Sign of f'
$x < -2$	increasing	+
$-2 < x < 1$	decreasing	-
$1 < x < 5$	increasing	+
$x > 5$	decreasing	-

The derivative crosses the x -axis at each turning point.

A possible graph of the derivative



Only the zeros and signs are fixed by the original graph. The vertical scale is not determined.

Exercise 3: Choosing differentiation rules

Exercise 3

Differentiate the following functions and keep their natural domains:

$$(a) \quad f(x) = \frac{x^2 + 3x}{x^3 + 1},$$

$$(b) \quad g(x) = x \ln^2(3x + 1),$$

$$(c) \quad h(x) = \arctan x.$$

Before calculating, identify the outermost operation in each expression.

The expression determines the first rule

Outermost operation	First rule
(a) quotient	quotient rule
(b) product	product rule, then chain rule
(c) inverse function	inverse-function rule

The domain belongs to the answer, especially in parts (a) and (b).

Part (a): prepare the quotient rule

Let

$$u(x) = x^2 + 3x, \quad v(x) = x^3 + 1.$$

Then

$$u'(x) = 2x + 3, \quad v'(x) = 3x^2.$$

Since $x^3 + 1 = 0$ only when $x = -1$,

$$\text{Dom}(f) = \mathbb{R} \setminus \{-1\}.$$

Part (a): apply and simplify

The quotient rule gives

$$\begin{aligned}f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} \\&= \frac{(2x+3)(x^3+1) - (x^2+3x)(3x^2)}{(x^3+1)^2} \\&= \frac{-x^4 - 6x^3 + 2x + 3}{(x^3+1)^2}.\end{aligned}$$

$$f'(x) = \frac{-x^4 - 6x^3 + 2x + 3}{(x^3 + 1)^2}, \quad x \neq -1.$$

Part (b): the product contains a composition

For

$$g(x) = x \ln^2(3x + 1),$$

the logarithm requires

$$3x + 1 > 0, \quad \text{so} \quad x > -\frac{1}{3}.$$

Write

$$g(x) = x [\ln(3x + 1)]^2.$$

The first step uses the product rule. The second factor then requires the chain rule twice.

Part (b): one factor for each layer

First,

$$\frac{d}{dx} [\ln(3x + 1)]^2 = 2 \ln(3x + 1) \frac{3}{3x + 1}.$$

Therefore,

$$\begin{aligned} g'(x) &= \ln^2(3x + 1) + x \left(2 \ln(3x + 1) \frac{3}{3x + 1} \right) \\ &= \ln^2(3x + 1) + \frac{6x \ln(3x + 1)}{3x + 1}. \end{aligned}$$

$$g'(x) = \ln^2(3x + 1) + \frac{6x \ln(3x + 1)}{3x + 1}, \quad x > -1/3.$$

Part (c): use the inverse identity

Let

$$y = \arctan x.$$

Then

$$\tan y = x.$$

Differentiating with respect to x gives

$$\frac{1}{\cos^2 y} \frac{dy}{dx} = 1,$$

and hence

$$\frac{dy}{dx} = \cos^2 y.$$

Part (c): return to the variable x

Since

$$1 + \tan^2 y = \frac{1}{\cos^2 y},$$

we obtain

$$\cos^2 y = \frac{1}{1 + \tan^2 y}.$$

Using $\tan y = x$,

$$\cos^2 y = \frac{1}{1 + x^2}.$$

$$(\arctan x)' = \frac{1}{1 + x^2} \text{ for every } x \in \mathbb{R}.$$

Exercise 4: Differentiability at the origin

Exercise 4

Consider

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine whether each function is differentiable at 0.

Both functions are continuous at the origin

For g ,

$$|x \sin(1/x)| \leq |x| \longrightarrow 0.$$

For h ,

$$|x^2 \sin(1/x)| \leq x^2 \longrightarrow 0.$$

Thus,

$$\lim_{x \rightarrow 0} g(x) = g(0) = 0, \quad \lim_{x \rightarrow 0} h(x) = h(0) = 0.$$

Continuity is necessary for differentiability, but it does not settle the question.

The difference quotient for g still oscillates

At the origin,

$$\frac{g(x) - g(0)}{x - 0} = \sin\left(\frac{1}{x}\right).$$

Choose

$$x_n = \frac{1}{\pi/2 + 2\pi n}, \quad y_n = \frac{1}{3\pi/2 + 2\pi n}.$$

Then $x_n, y_n \rightarrow 0$, but

$$\sin(1/x_n) = 1, \quad \sin(1/y_n) = -1.$$

$g'(0)$ does not exist.

The extra factor x makes h differentiable

For h ,

$$\frac{h(x) - h(0)}{x - 0} = x \sin\left(\frac{1}{x}\right).$$

Since

$$|x \sin(1/x)| \leq |x| \longrightarrow 0,$$

the squeeze theorem gives

$$\lim_{x \rightarrow 0} \frac{h(x) - h(0)}{x - 0} = 0.$$

$$h'(0) = 0.$$

The two functions differ by one power of x

Function	Continuous at 0	Differentiable at 0
$x \sin(1/x)$	yes	no
$x^2 \sin(1/x)$	yes	yes, with derivative 0

The function may approach zero while its difference quotient continues to oscillate.

Exercise 5: A constrained tangent line

Exercise 5

Among all lines passing through the origin, find the one that is tangent to

$$f(x) = \ln x.$$

The point of tangency is unknown. Denote it by

$$(a, \ln a), \quad a > 0.$$

Write the tangent line at an arbitrary point

Since

$$f'(a) = \frac{1}{a},$$

the tangent line at $(a, \ln a)$ is

$$y - \ln a = \frac{1}{a}(x - a).$$

After simplifying,

$$y = \frac{x}{a} + \ln a - 1.$$

The condition that the line passes through the origin will determine a .

The origin determines the point of tangency

Substituting $(x, y) = (0, 0)$ into

$$y = \frac{x}{a} + \ln a - 1$$

gives

$$0 = \ln a - 1.$$

Therefore,

$$\ln a = 1, \quad a = e.$$

The slope is $f'(e) = 1/e$.

The required tangent line

At $a = e$,

$$(a, \ln a) = (e, 1).$$

Thus,

$$y - 1 = \frac{1}{e}(x - e),$$

so

$$\boxed{y = \frac{x}{e}}.$$

