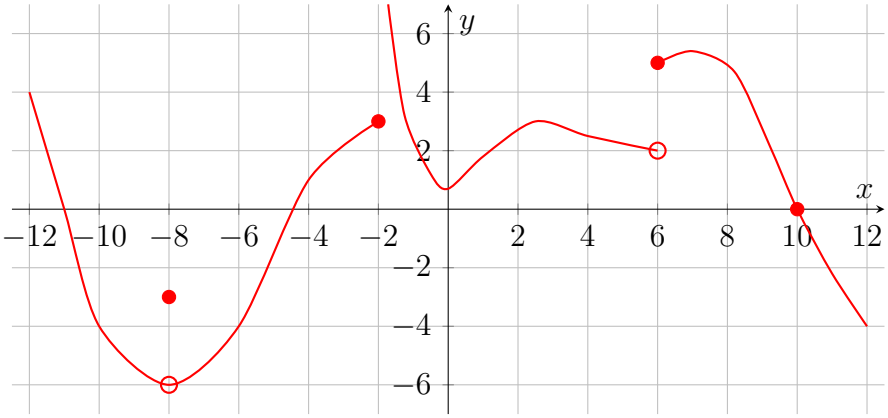
Calculus 1

Week 3[1]

Problem 1. The graph of a function f is shown below. Determine the points at which f is not continuous and classify each discontinuity. At a discontinuity, also decide whether the function is continuous from the left or from the right.



Solution. The only points requiring attention are $x = -8$, $x = -2$ and $x = 6$.

At $x = -8$, the curve approaches -6 from both sides, but the filled point is at height -3. Thus,

$$\lim_{x \to -8} f(x) = -6 \neq -3 = f(-8).$$

This is a removable discontinuity. The function is neither left-continuous nor right-continuous there because neither one-sided limit equals $f(-8)$.

At $x = -2$,

$$\lim_{x \to -2^-} f(x) = 3 = f(-2), \qquad \lim_{x \to -2^+} f(x) = +\infty.$$

The two-sided limit does not exist. Hence, f is continuous from the left at -2, but it is not continuous from the right.

Finally,

$$\lim_{x \to 6^-} f(x) = 2, \qquad \lim_{x \to 6^+} f(x) = 5 = f(6).$$

This is a jump discontinuity. The function is continuous from the right at 6, but not from the left. It is continuous at every other interior point displayed in the graph.

Problem 2. Discuss the continuity of the function

$$g(x) = \begin{cases} 1 - 3x, & x < -6, \\ 7, & x = -6, \\ x^3, & -6 < x < 1, \\ 1, & x = 1, \\ 2 - x, & x > 1. \end{cases}$$

[1]Since one needs to remind constantly the previous contents as they will be important afterwards, there are some exercises in the present list about continuity and limits as well.

Solution. Each formula is continuous in the interior of the interval on which it is used. We only need to study the junction points -6 and 1.

At $x = -6$,

$$\lim_{x\to -6^-} g(x) = 1 - 3(-6) = 19,$$
$$\lim_{x\to -6^+} g(x) = (-6)^3 = -216.$$

The one-sided limits are different, so the two-sided limit does not exist. In addition, neither one-sided limit equals $g(-6) = 7$. Therefore, g is neither left-continuous nor right-continuous at -6.

At $x = 1$,

$$\lim_{x\to 1^-} g(x) = 1, \qquad \lim_{x\to 1^+} g(x) = 2 - 1 = 1, \qquad g(1) = 1.$$

Thus, g is continuous at 1. Consequently, g is continuous on

$$\mathbb{R} \setminus \{-6\}.$$

Problem 3. Discuss the continuity of

$$f(x) = \begin{cases} x + e^x, & x < 0, \\ x^2, & x \geqslant 0. \end{cases}$$

Solution. The function is continuous on $(-\infty, 0)$ and on $(0, +\infty)$ because each formula is a sum of continuous functions. At the junction point,

$$\lim_{x\to 0^-} f(x) = 0 + e^0 = 1, \qquad \lim_{x\to 0^+} f(x) = 0, \qquad f(0) = 0.$$

The one-sided limits are different, so f has a jump discontinuity at 0. Since the right-hand limit equals $f(0)$, the function is continuous from the right at 0, but not from the left.

Problem 4. Find the domain, calculate the limits at the boundary points and discuss the continuity of

$$f(x) = \begin{cases} \arcsin\left(\dfrac{x+1}{2x}\right), & x > 1, \\ 2x, & x \leqslant 1. \end{cases}$$

Solution. The second formula is defined for every $x \leqslant 1$. If $x > 1$, then

$$\frac{x+1}{2x} = \frac{1}{2} + \frac{1}{2x},$$

and therefore

$$\frac{1}{2} < \frac{x+1}{2x} < 1.$$

The argument of arcsin belongs to $[-1, 1]$, so

$$\mathrm{Dom}(f) = \mathbb{R}.$$

At the infinite boundary points,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x = -\infty$$

and

$$\lim_{x\to+\infty} f(x) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

The formulas are continuous away from $x = 1$. At that point,

$$\lim_{x\to1^-} f(x) = 2 = f(1),$$

whereas

$$\lim_{x\to1^+} f(x) = \arcsin(1) = \frac{\pi}{2}.$$

Since $2 \neq \pi/2$, the function has a jump discontinuity at 1. It is continuous from the left there and is continuous on $\mathbb{R} \setminus \{1\}$.

Problem 5. Use the definition of the derivative to calculate the derivatives of

(a) $f(x) = x^2$;

(b) $g(x) = \sin x$.

Solution. **(a)** At an arbitrary point $a \in \mathbb{R}$,

$$\begin{aligned} f'(a) &= \lim_{h\to0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to0} \frac{2ah + h^2}{h} = \lim_{h\to0}(2a + h) = 2a. \end{aligned}$$

Since a was arbitrary, $f'(x) = 2x$ for every $x \in \mathbb{R}$.

(b) Using the addition formula for sine,

$$\begin{aligned} g'(a) &= \lim_{h\to0} \frac{\sin(a+h) - \sin a}{h} \\ &= \lim_{h\to0} \left(\sin a \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h}\right). \end{aligned}$$

The standard limits

$$\lim_{h\to0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to0} \frac{\cos h - 1}{h} = 0$$

give

$$g'(a) = \cos a.$$

Therefore, $(\sin x)' = \cos x$ for every $x \in \mathbb{R}$.

Problem 6. Determine whether $f'(0)$ exists for $f(x) = |x|$.

Solution. Since $f(0) = 0$, the right-hand difference quotient is

$$\lim_{h \to 0^+} \frac{|h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

On the other hand,

$$\lim_{h \to 0^-} \frac{|h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

The one-sided derivatives are different. Consequently, $f'(0)$ does not exist, even though f is continuous at 0.

Problem 7. Differentiate the following functions:

(a) $f(x) = x^2 \sin x + 2$;

(b) $h(x) = \dfrac{\sin x}{x}$;

(c) $g(x) = 2$.

Solution. **(a)** The product rule gives $f'(x) = 2x \sin x + x^2 \cos x$. The derivative of the constant term is zero.

(b) The natural domain is $\mathbb{R} \setminus \{0\}$. By the quotient rule,

$$h'(x) = \frac{x \cos x - \sin x}{x^2}, \qquad x \neq 0.$$

(c) A constant function has derivative zero. Indeed,

$$g'(x) = \lim_{h \to 0} \frac{2 - 2}{h} = 0.$$

Problem 8. Use the chain rule to differentiate

(a) $f(x) = \sin(x^2)$;

(b) $g(x) = \sin^2 x$.

Solution. **(a)** The outer function is $\sin u$ and the inner function is $u = x^2$. Thus,

$$f'(x) = \cos(x^2) \cdot 2x = 2x \cos(x^2).$$

(b) We write $g(x) = (\sin x)^2$. Differentiating the outer square and then the inner sine gives $g'(x) = 2 \sin x \cos x$.

Problem 9. Use the inverse-function rule to prove that

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}.$$

Solution. Let $y = \arctan x$. Then $y \in (-\pi/2, \pi/2)$ and $\tan y = x$. Differentiating both sides with respect to x gives

$$\frac{1}{\cos^2 y}\frac{dy}{dx} = 1.$$

Therefore,

$$\frac{dy}{dx} = \cos^2 y.$$

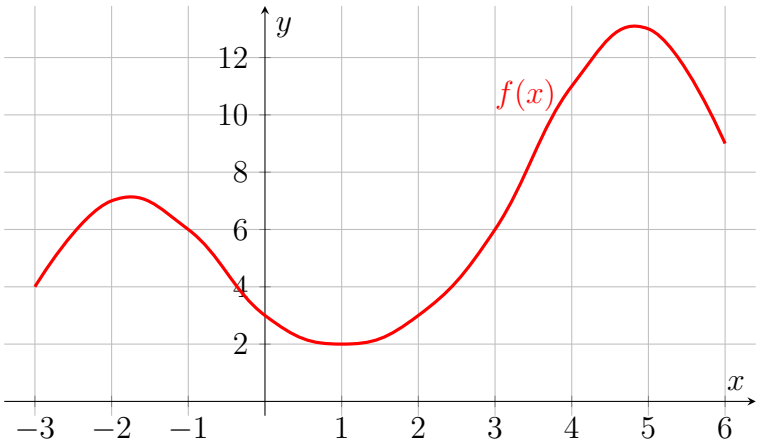
Since $1 + \tan^2 y = 1/\cos^2 y$ and $\tan y = x$, we obtain

$$\cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

Hence,

$$(\arctan x)' = \frac{1}{1 + x^2}, \qquad x \in \mathbb{R}.$$

Problem 10. The graph of a differentiable function f is shown below. Sketch a possible graph of f'.



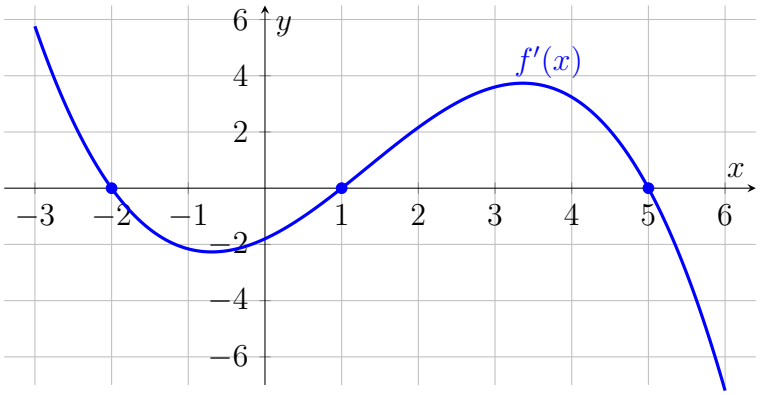
Solution. The graph has horizontal tangents at the local maximum $x = -2$, the local minimum $x = 1$ and the local maximum $x = 5$. Therefore,

$$f'(-2) = f'(1) = f'(5) = 0.$$

Moreover,

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, 5)$	5	$(5, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$

A possible qualitative sketch is the following.



Only the zeros and the signs of this sketch are determined by the information in the original graph; its exact vertical scale is not determined.

Problem 11. Find the natural domain and derivative of

$$f(x) = x^3 + \sqrt{x} + \frac{1}{x^2} - 3.$$

Solution. The square root requires $x \geqslant 0$, while the term $1/x^2$ requires $x \neq 0$. Thus,

$$\mathrm{Dom}(f) = (0, +\infty).$$

On this interval, we may write

$$f(x) = x^3 + x^{1/2} + x^{-2} - 3.$$

Term-by-term differentiation gives

$$f'(x) = 3x^2 + \frac{1}{2\sqrt{x}} - \frac{2}{x^3}, \qquad x > 0.$$

Problem 12. Calculate the following limits:

(a) $\lim_{x\to+\infty} \dfrac{\cos x}{x^2+1}$;

(b) $\lim_{x\to+\infty} e^x \sin(2x)$;

(c) $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$;

(d) $\lim_{x\to-\infty} \dfrac{1/e^x}{2+\cos x}$.

Solution. **(a)** Since $|\cos x| \leqslant 1$,

$$\left|\frac{\cos x}{x^2+1}\right| \leqslant \frac{1}{x^2+1}.$$

The right-hand side tends to zero, so the squeeze theorem gives

$$\lim_{x\to+\infty}\frac{\cos x}{x^2+1}=0.$$

(b) This limit does not exist. Indeed, let

$$x_n=\frac{\pi}{4}+\pi n,\qquad y_n=\frac{3\pi}{4}+\pi n.$$

Then $x_n, y_n\to+\infty$, while

$$\sin(2x_n)=1,\qquad \sin(2y_n)=-1.$$

Consequently,

$$e^{x_n}\sin(2x_n)=e^{x_n}\to+\infty,$$

whereas

$$e^{y_n}\sin(2y_n)=-e^{y_n}\to-\infty.$$

The function has different behaviour along two sequences tending to $+\infty$.

(c) We have

$$|x\sin(1/x)|\leqslant|x|\,.$$

Therefore, the squeeze theorem yields

$$\lim_{x\to0}x\sin\left(\frac{1}{x}\right)=0.$$

(d) Since $1\leqslant 2+\cos x\leqslant 3$, we have

$$\frac{e^{-x}}{3}\leqslant\frac{1/e^x}{2+\cos x}\leqslant e^{-x}.$$

As $x\to-\infty$, the lower bound $e^{-x}/3$ tends to $+\infty$. Hence,

$$\lim_{x\to-\infty}\frac{1/e^x}{2+\cos x}=+\infty.$$

Problem 13. Consider the functions

$$g(x)=\begin{cases}x\sin(1/x), & x\neq0,\\ 0, & x=0,\end{cases}$$

and

$$h(x)=\begin{cases}x^2\sin(1/x), & x\neq0,\\ 0, & x=0.\end{cases}$$

Determine whether each function is differentiable at 0.

Solution. For g, the difference quotient at zero is

$$\frac{g(x)-g(0)}{x-0}=\sin\left(\frac{1}{x}\right),\qquad x\neq0.$$

This expression has no limit as $x \to 0$. Therefore, $g'(0)$ does not exist. Notice that g is nevertheless continuous at 0 because

$$|x \sin(1/x)| \leqslant |x| \longrightarrow 0.$$

For h,

$$\frac{h(x) - h(0)}{x - 0} = x \sin\left(\frac{1}{x}\right).$$

Since

$$|x \sin(1/x)| \leqslant |x| \longrightarrow 0,$$

the derivative exists and $h'(0) = 0$.

Problem 14. Find the derivatives of the following functions. In each case, respect the natural domain of the original function.

(a) $f(x) = \dfrac{x^2 + 3x}{x^3 + 1}$;

(b) $f(x) = e^x \cos x$;

(c) $f(x) = \sqrt{x + 2}$;

(d) $f(x) = \dfrac{e^x}{\sin x}$;

(e) $f(x) = \ln(x^3 + 1)$;

(f) $f(x) = (x^2 + 1) \ln x$;

(g) $f(x) = x \ln^2(3x + 1)$;

(h) $f(x) = \dfrac{1}{1 - \cos x}$;

(i) $f(x) = x^2 e^x \sin x$;

(j) $f(x) = \dfrac{\ln(x^4 + \sqrt{x})}{\arctan x}$.

Solution. **(a)** The denominator vanishes only at $x = -1$. The quotient rule gives

$$f'(x) = \frac{(2x + 3)(x^3 + 1) - (x^2 + 3x)(3x^2)}{(x^3 + 1)^2} = \frac{-x^4 - 6x^3 + 2x + 3}{(x^3 + 1)^2}, x \neq -1.$$

(b) By the product rule,

$$f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$$

(c) The original function has domain $[-2, +\infty)$. At every interior point of this domain,

$$f'(x) = \frac{1}{2\sqrt{x + 2}}, \qquad x > -2.$$

There is no finite two-sided derivative at the endpoint $x = -2$.

(d) The domain is $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. On this domain,

$$f'(x) = \frac{e^x \sin x - e^x \cos x}{\sin^2 x} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}.$$

(e) The logarithm requires $x^3 + 1 > 0$, so $x > -1$. The chain rule gives

$$f'(x) = \frac{3x^2}{x^3 + 1}, \qquad x > -1.$$

(f) The domain is $(0, +\infty)$. Using the product rule,

$$f'(x) = 2x \ln x + \frac{x^2 + 1}{x}, \qquad x > 0.$$

(g) The logarithm requires $3x + 1 > 0$, so $x > -1/3$. We obtain

$$f'(x) = \ln^2(3x + 1) + x \cdot 2 \ln(3x + 1) \frac{3}{3x + 1} = \ln^2(3x + 1) + \frac{6x \ln(3x + 1)}{3x + 1}.$$

(h) The denominator vanishes when $\cos x = 1$, that is, at $x = 2k\pi$. Thus, $\mathrm{Dom}(f) = \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$. Since $(1 - \cos x)' = \sin x$,

$$f'(x) = -\frac{\sin x}{(1 - \cos x)^2}$$

on the same domain.

(i) Applying the product rule twice gives

$$f'(x) = 2xe^x \sin x + x^2 e^x \sin x + x^2 e^x \cos x = xe^x \Big((x + 2) \sin x + x \cos x \Big).$$

(j) The square root requires $x \geqslant 0$. At $x = 0$, the argument of the logarithm is zero, so the logarithm is not defined. For $x > 0$, both $x^4 + \sqrt{x}$ and $\arctan x$ are positive. Hence, $\mathrm{Dom}(f) = (0, +\infty)$. Set

$$u(x) = \ln(x^4 + \sqrt{x}), \qquad v(x) = \arctan x.$$

Then

$$u'(x) = \frac{4x^3 + 1/(2\sqrt{x})}{x^4 + \sqrt{x}}, \qquad v'(x) = \frac{1}{1 + x^2}.$$

Therefore,

$$f'(x) = \frac{\dfrac{4x^3 + 1/(2\sqrt{x})}{x^4 + \sqrt{x}} \arctan x - \dfrac{\ln(x^4 + \sqrt{x})}{1 + x^2}}{(\arctan x)^2}, \qquad x > 0.$$

Problem 15. Find the equation of the tangent line to

$$f(t) = 4t - 8\sqrt{t}$$

at $t = 16$.

Solution. First, $f(16) = 4 \cdot 16 - 8\sqrt{16} = 64 - 32 = 32$. For $t > 0$, $f'(t) = 4 - \frac{4}{\sqrt{t}}$. Thus,

$$f'(16) = 4 - \frac{4}{4} = 3.$$

The point of tangency is $(16, 32)$ and the slope is 3. Therefore, $y - 32 = 3(t - 16)$ or, equivalently, $y = 3t - 16$.

Problem 16. Among all lines passing through the origin, find the one that is tangent to the graph of $f(x) = \ln x$.

Solution. Let the tangent touch the graph at $x = a > 0$. Since $f'(a) = 1/a$, its equation is

$$y - \ln a = \frac{1}{a}(x - a).$$

Equivalently,

$$y = \frac{x}{a} + \ln a - 1.$$

The line passes through $(0, 0)$ precisely when $0 = \ln a - 1$. Thus, $\ln a = 1$ and $a = e$. The required tangent line is $y = \frac{x}{e}$.

Problem 17. Find the domain, calculate the limits at the boundary points and discuss the continuity of

$$f(x) = \frac{1 - \sqrt{1 + \sin x}}{x}.$$

Solution. Since $-1 \leqslant \sin x \leqslant 1$, we have

$$1 + \sin x \geqslant 0$$

for every real x. The only restriction comes from the denominator. Therefore,

$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty).$$

At $x = 0$, direct substitution gives the indeterminate form $0/0$. We rationalize:

$$\frac{1 - \sqrt{1 + \sin x}}{x} = \frac{(1 - \sqrt{1 + \sin x})(1 + \sqrt{1 + \sin x})}{x(1 + \sqrt{1 + \sin x})} = -\frac{\sin x}{x(1 + \sqrt{1 + \sin x})}.$$

Consequently,

$$\lim_{x \to 0} f(x) = -\frac{1}{1 + 1} = -\frac{1}{2}.$$

The numerator is bounded because

$$0 \leqslant \sqrt{1 + \sin x} \leqslant \sqrt{2}.$$

Hence,

$$\lim_{x \to -\infty} f(x) = 0, \qquad \lim_{x \to +\infty} f(x) = 0.$$

The function is continuous at every point of its domain. The point $x = 0$ is a removable discontinuity, and defining $f(0) = -1/2$ would give a continuous extension to all of $\mathbb{R}$.