

A strictly convex and smooth Banach space with the Daugavet property

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This talk is based on a joint work with

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DANTAS-KAASIK-PERREAU (2026), [arXiv:2609.12562](https://arxiv.org/abs/2609.12562).

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- ★ $\|f\|_1$: the usual L_1 -norm.
- ★ $\|f\|$: the equivalent norm constructed in this talk.

Definition

A Banach space X has the **Daugavet property** (DPr, for short) if

$$\|Id_X + T\| = 1 + \|T\|$$

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Daugavet proved this identity for every compact operator on $C[0, 1]$.

DAUGAVET (1963).

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- ★ $\mathcal{F}(M)$ when M is a complete length space.

DAUGAVET (1963)

KADETS-SHVIDKOY-SIROTKIN-WERNER (2000)

GARCÍA-LIROLA-PROCHÁZKA-RUEDA ZOCA (2018)

For $x^* \in S_{X^*}$ and $\alpha > 0$, a **slice** is

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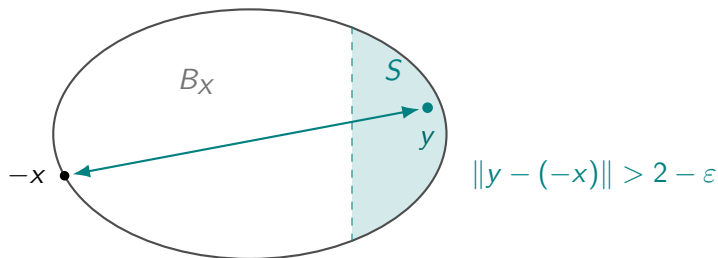
Equivalent formulation of the DPr

For every $x \in S_X$, every slice $S \subset B_X$, and every $\varepsilon > 0$, there exists $y \in S$ such that

$$\|x + y\| > 2 - \varepsilon.$$

KADETS-SHVIDKOY-SIROTKIN-WERNER (2000)

Every slice reaches almost distance two



Every slice contains a point almost diametral to $-x$.

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The Daugavet property forces

- ★ every slice of B_X to have diameter two
- ★ and, in particular, finite-dimensional spaces cannot have the DPr.
(finite-dimensional unit balls have arbitrarily small slices!)
- ★ many isometric copies of ℓ_1 ,
- ★ failure of the Radon–Nikodým property.

Moreover, X^* is neither strictly convex nor smooth.

KADETS-SHVIDKOY-SIROTKIN-WERNER (2000).

KADETS-MARTÍN-MERÍ-PAYÁ (2009)

Strict convexity (SC, for short)

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$$\left\| \frac{x + y}{2} \right\| < 1.$$

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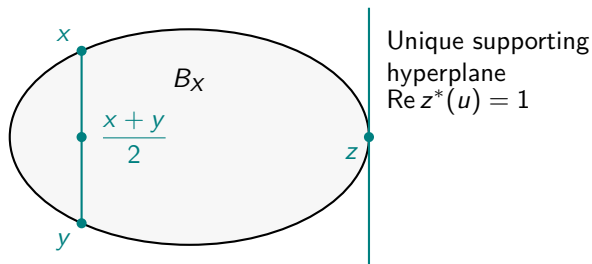
$$\left\| \frac{x + y}{2} \right\| < 1.$$

Gâteaux smoothness (smooth, for short)

For every $x \in S_X$ and every $h \in X$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t}$$

exists, with $t \in \mathbb{R}$.



Strict convexity

For all distinct $x, y \in S_X$,

$$\left\| \frac{x+y}{2} \right\| < 1.$$

Gâteaux smoothness

For every $z \in S_X$, there is a unique $z^* \in S_{X^*}$ such that

$$z^*(z) = 1.$$

Kadets (1996)

- ★ Does there exist a **strictly convex Banach space** with the DPr?
- ★ Does there exist a **smooth Banach space** with the DPr?

KADETS (1996)

KADETS-MARTÍN-RUEDA ZOCA-WERNER (2025), Questions 3.3 and 3.4

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★ **The Cobollo-Hájek renorming gave us the most hope.**

ABRAHAMSEN-HÁJEK-NYGAARD-TALPONEN-TROYANSKI (2016)

ABRAHAMSEN-HÁJEK-LIMA-TROYANSKI (2026)

NYGAARD-PÖLDVERE-TROYANSKI-VIIL (2024)

COBOLLO-HÁJEK (2025)

DE BERNARDI-PRETI-SOMAGLIA (2025)

The renormings we ruled out

Renorming	DPr	Daugavet pts	Octahedral
De Bernardi-Preti-Somaglia	✗	✗	✗
De Bernardi-Preti-Somaglia (modified CH)	✗	✗	✓
De Bernardi-Somaglia	✗	✗	✗
Abrahamsen-Hájek-Lima-Troyanski	✗	✗	✗
Abrahamsen-Hájek-Nygaard-Talponen-Troyanski	✗	✗	✗
Nygaard-Pöldvere-Troyanski-Viil	✗	?	✗
Cobollo-Hájek	✗	?	✓
Hilbertian ℓ_r -sum, $1 < r < \infty$	✗	✗	✗

The last row concerns equivalent norms of the form

$$\|x\|_{\text{new}} = \left(p(x)^r + \|Ax\|_H^r \right)^{1/r}, \quad 1 < r < \infty,$$

where p is a seminorm, H is a Hilbert space, and $A : X \rightarrow H$ is a nonzero bounded linear operator.

Definition

A real Banach space Y is **octahedral** (OH, for short) if, for every finite-dimensional $E \subset Y$ and every $\varepsilon > 0$, there exists $z \in S_Y$ such that

$$\|x + tz\| \geq (1 - \varepsilon)(\|x\| + |t|)$$

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The DPr implies OH, but the converse fails.

Example. ℓ_1 is OH but does not have the DPr. Indeed, the rank-one projection $P(x) = x_1 e_1$ satisfies

$$\|\text{Id} - P\| = 1 < 2 = 1 + \|P\|.$$

KADETS-SHVIDKOY-SIROTKIN-WERNER (2000)

Cobollo-Hájek (2025)

If a real Banach space Y admits a Gâteaux smooth equivalent norm and contains a complemented copy of ℓ_1 , then Y admits an equivalent norm which is **smooth and OH**.

COBOLLO-HÁJEK, 2025.

Write $Y = Y_0 \oplus \ell_1$ and set

$$Y_n = Y_0 + \text{span}\{e_1, \dots, e_n\}.$$

Starting with a smooth norm on Y_0 , define the next unit ball by

$$B_n = \left\{ x + te_n : \begin{array}{l} x \in Y_{n-1}, \quad |t| \leq 1, \\ \|x\|_{n-1} \leq 1 - f_n(|t|) \end{array} \right\}.$$

The convex profiles f_n are flat near 0, almost linear in the middle and rounded near 1.

Cobollo-Hájek (2025)

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- ★ New coordinates give asymptotic ℓ_1 behavior.
- ★ Directional estimates preserve smoothness on all of Y .

COBOLLO-HÁJEK (2025).

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- ★ The obstacle was **completeness**.
- ★ Obtaining the desired geometry on an increasing union did not give us control of smoothness or strict convexity after completion.
- ★ Cobollo-Hájek's construction seems not to give a complete space.

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Strict convexity and the Daugavet property

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Smoothness and the Daugavet property

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The question concerned **Banach spaces**!

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- ★ After several days, **ChatGPT suggested the norm** used in our paper.
- ★ The solution appeared to involved tools from Dynamical Systems.
- ★ We took the norm as a starting point, developed our own proofs and obtained stronger results than the ones it suggested.

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7. Ask AI to check your proofs, then assess its feedback.
8. Repeat 1-7 until you are happy with the results.
9. Write a paper where your community understands everything with clear motivation and clear context citing previous works.

CONSTRUCTING THE NORM

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Choose $a_k > 0$ for $k \in \mathbb{Z}$ such that

$$\sum_{k \in \mathbb{Z}} a_k = 1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \sqrt{a_k} < \infty.$$

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$$X := (L_1(\mu), \|\cdot\|).$$

An equivalent norm on all of $L_1(\mu)$

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Thus X is **complete**.

For every $\varepsilon > 0$, the weights can be chosen so that

$$\|f\|_1 \leq \|f\| \leq (1 + \varepsilon) \|f\|_1.$$

Definition 3.2

The transformation T is **ergodic** if, for every $A \in \Sigma$,

$$\mu(A \Delta T^{-1}A) = 0 \implies \mu(A) \in \{0, 1\}.$$

Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

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Example. On $[0, 1)$ with its Borel σ -algebra and Lebesgue measure, fix $\theta \in (0, 1) \setminus \mathbb{Q}$ and define

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Then T is **invertible, measure-preserving and ergodic**.

Fact [D. Kaasik, Perreau]

If T is ergodic and $\mu(A) > 0$, then

$$\mu \left(\bigcup_{k \in \mathbb{Z}} T^{-k} A \right) = 1.$$

Theorem [D. Kaasik, Perreau]

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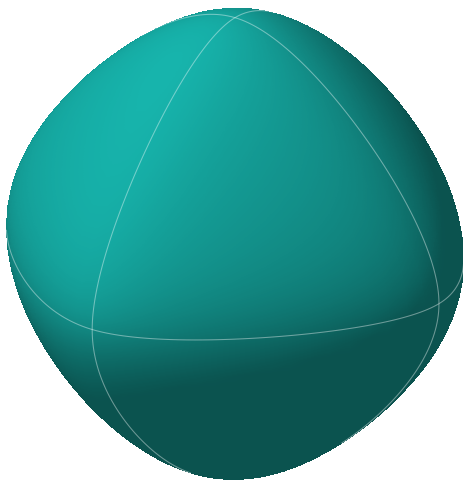
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1. T is ergodic.
2. $\|\cdot\|$ is strictly convex.
3. $\|\cdot\|$ is Gâteaux smooth.
4. X does not contain an isometric copy of ℓ_1^2 .



The “3-dimensional” unit ball $B_{(X, \|\cdot\|)}$ of our space

Definition

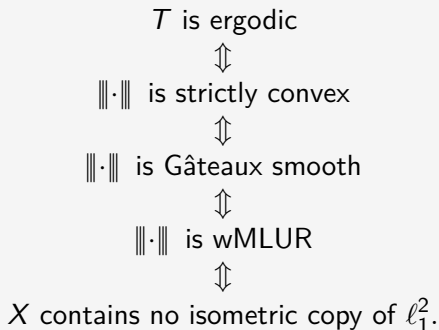
A Banach space Y is **wMLUR** if, whenever $x \in S_Y$ and $(h_n) \subset Y$ satisfy

$$\|x + h_n\| \longrightarrow 1, \quad \|x - h_n\| \longrightarrow 1,$$

we have $h_n \xrightarrow{w} 0$.

Every wMLUR norm is strictly convex.

Theorem [D. Kaasik and Perreau]



Theorem [D. Kaasik and Perreau]

Let (Ω, Σ, μ) be an atomless probability space and $T : \Omega \rightarrow \Omega$ be an invertible measure-preserving transformation. Then

$$X = (L_1(\mu), \|\cdot\|)$$

has the Daugavet property.

★ **Ergodicity is not needed here.**

For $m \in \mathbb{N}$, put

$$J_m f(\omega) = (\sqrt{a_k} f(T^k \omega))_{|k| \leq m},$$

$$\|f\|_m = \int_{\Omega} \|J_m f(\omega)\|_2 d\mu.$$

Tail estimate [D. Kaasik and Perreau]

$$0 \leq \|f\| - \|f\|_m \leq \left(\sum_{|k| > m} \sqrt{a_k} \right) \|f\|_1.$$

The convergence is uniform on bounded subsets of X .

Lemma [D., Kaasik and Perreau]

For every $h \in S_X$ and $\rho, \eta > 0$, there exist $h_1, \dots, h_q \in L_1(\mu)$ such that

$$(a) \quad h = \sum_{i=1}^q h_i,$$

$$(b) \quad \mu(\text{supp } h_i) < \rho \text{ for every } 1 \leq i \leq q, \text{ and}$$

$$(c) \quad \sum_{i=1}^q \|h_i\| < 1 + \eta.$$

We split h into pieces with **small supports**, while keeping the sum of their norms arbitrarily close to $\|h\| = 1$.

Recall that throughout this proof, μ is atomless.

Lemma [D., Kaasik and Perreau]

Let $\Phi \in S_{X^*}$, $0 < \alpha < 1$ and $\rho > 0$. There exists $y \in S_X$ such that

$$\operatorname{Re} \Phi(y) > 1 - \alpha, \quad \mu(\operatorname{supp} y) < \rho.$$

★ Every slice of B_X contains a **unit vector supported on a set of arbitrarily small measure**.

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Lemma [D., Kaasik and Perreau]

If $(g_n) \subset S_X$ and $\mu(\operatorname{supp} g_n) \rightarrow 0$, then

$$\lim_n \|f + g_n\| = \|f\| + 1$$

for every $f \in L_1(\mu)$.

★ Adding a unit vector with sufficiently small support to a **fixed** f increases its norm by almost 1.

Theorem [D., Kaasik and Perreau]

$X = (L_1(\mu), \|\cdot\|)$ has the DPr.

Recall: the slice characterization

X has the DPr $\Leftrightarrow$ for every $f \in S_X$, every slice $S \subset B_X$ and every $\varepsilon > 0$, there exists $g \in S$ such that

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- ★ For large n , $g_n \in S$ and $\|f + g_n\| > 2 - \varepsilon$.
- ★ The slice characterization gives the DPr.

□

Definition. Y has the **ODPr** if, for every $x_1, \dots, x_n \in S_Y$, every slice $S \subset B_Y$, and every $\varepsilon > 0$, there exists $y \in S$ such that

$$\forall z \in B_Y \quad \exists R \in \mathcal{L}(Y) :$$

$$\|R\| \leq 1 + \varepsilon \quad \|Rx_i - x_i\| < \varepsilon, \quad \forall 1 \leq i \leq n \quad \text{and} \quad Ry = z.$$

RUEDA ZOCA-TRADACETE-VILLANUEVA (2021)

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Theorem [D. Kaasik and Perreau]

If μ is atomless and T is invertible and measure-preserving, then $X = (L_1(\mu), \|\cdot\|)$ has the ODPr.

ODPr implies DPr. The converse fails in general.

HAMAD (2026).

Theorem [D. Kaasik and Perreau]

For atomless μ and invertible measure-preserving T ,

$$X^* = (L_\infty(\mu), \|\cdot\|_*)$$

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Fact [D. Kaasik and Perreau]

The bidual X^{**} **fails** the DPr.

Open problem

Does there exist a Banach space Y such that Y^{**} has the DPr?

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★ **A candidate we are investigating right at this moment:** there exists a complete metric space M such that the Lipschitz-free space $\mathcal{F}(M)$ is such that $\mathcal{F}(M)^{**}$ satisfies the DPr.

Midpoint local uniform rotundity

A Banach space Y is **MLUR** if

$$x \in S_Y, \quad \|x \pm h_n\| \rightarrow 1 \quad \implies \quad \|h_n\| \rightarrow 0.$$

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- ★ wMLUR, but
- ★ **not** MLUR.
- ★ Since DPr implies OH and OH rules out LUR points, our renorming **cannot** be LUR.

LUR $\Rightarrow$ MLUR $\Rightarrow$ wMLUR $\Rightarrow$ strict convexity.

Our equivalent norm on $L_1[0, 1]$

- ★ is **strictly convex** (in fact, **wMLUR**)
- ★ is **Gâteaux smooth**
- ★ and has the **Daugavet property**.

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- ★ is **strictly convex** (in fact, **wMLUR**)
- ★ is **Gâteaux smooth**
- ★ and has the **Daugavet property**.

Fréchet smoothness is **impossible** for a norm with the DPr.

KADETS-MARTÍN-MERÍ-PAYÁ (2009)

★ Can our norm have a smoothness property strictly between Gâteaux and Fréchet smoothness?

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- ★ Is this norm useful in a different context?
- ★ Would it solve another problem?

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DÍKY MOC!