

10 Lecture #7: Week 4

10.1 From local information to global information

The derivative is defined at one point, but its most useful consequences often concern an entire interval. This is the central idea of this lecture. If we know something about $f'(x)$ for every x between a and b , then we should be able to say something about the total change $f(b) - f(a)$. The Mean Value Theorem makes this connection precise.

Before stating it, we need one elementary observation about local extrema.

Definition 10.1. Let f be defined on an interval I and let $c \in I$.

- The function f has a **local maximum** at c if there is $\delta > 0$ such that

$$f(x) \leq f(c)$$

whenever $x \in I$ and $|x - c| < \delta$.

- The function f has a **local minimum** at c if there is $\delta > 0$ such that

$$f(c) \leq f(x)$$

whenever $x \in I$ and $|x - c| < \delta$.

A local maximum or minimum is called a **local extremum**.

The word “local” is important. It means that we compare $f(c)$ only with values of the function near c . A local maximum need not be the largest value of the function on its whole domain.

Theorem 10.2 (Fermat’s theorem). Let I be an interval, let c be an interior point of I , and suppose that $f : I \rightarrow \mathbb{R}$ is differentiable at c . If f has a local extremum at c , then

$$f'(c) = 0.$$

Proof. Suppose, for instance, that f has a local maximum at c . For $x < c$ sufficiently close to c , we have $f(x) - f(c) \leq 0$ and $x - c < 0$, so

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

Therefore, the left derivative, if it exists, is non-negative. For $x > c$ sufficiently close to c , the numerator is still non-positive but the denominator is positive, so

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

The right derivative is non-positive. Since $f'(c)$ exists, the two one-sided limits must be equal. The only possibility is $f'(c) = 0$. The proof for a local minimum is analogous. $\square$

Keep in mind that every hypothesis matters. The function $f(x) = |x|$ has a local minimum at 0, but it is not differentiable there. The function $f(x) = x$ on $[0, 1]$ has an absolute maximum at the endpoint 1, but $f'(1) = 1$ if the derivative is understood by extending the formula. Fermat’s theorem concerns an interior point at which the function is differentiable.

Definition 10.3. An interior point c of the domain of f is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's theorem says that differentiable local extrema can only occur at critical points. It does not say that every critical point is an extremum. For example, $f(x) = x^3$ satisfies $f'(0) = 0$, but f is strictly increasing through the origin.

10.2 Rolle's theorem

Suppose that a continuous curve starts and finishes at the same height. If it is smooth between the endpoints, then at some intermediate point its tangent must be horizontal. This geometric fact is Rolle's theorem.

Theorem 10.4 (Rolle's theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If

$$f(a) = f(b),$$

then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. By the Extreme Value Theorem, f attains both a maximum and a minimum on $[a, b]$. If the maximum and minimum are equal, then f is constant and every $c \in (a, b)$ satisfies $f'(c) = 0$. Otherwise, at least one of the two extreme values is attained at an interior point, because the endpoint values are equal. Fermat's theorem then gives $f'(c) = 0$ at that interior extremum. $\square$

Example 10.5. Let $f(x) = x^3 - x$ on $[-1, 1]$. We have $f(-1) = f(1) = 0$, and the hypotheses of Rolle's theorem are satisfied. Since

$$f'(x) = 3x^2 - 1,$$

the points supplied by the theorem are

$$c = -\frac{1}{\sqrt{3}} \quad \text{and} \quad c = \frac{1}{\sqrt{3}}.$$

Rolle's theorem guarantees at least one point, but there may be more than one.

Rolle's theorem is also useful for counting zeros. If a differentiable function has two distinct zeros, then its derivative has a zero between them. More generally, if f has m distinct zeros, repeated use of Rolle's theorem shows that f' has at least $m - 1$ distinct zeros.

Corollary 10.6. Let f be continuous on an interval and differentiable in its interior. If f' has at most k zeros, then f has at most $k + 1$ zeros.

10.3 The Mean Value Theorem

The average rate of change of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

It is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem says that, at some point between a and b , the instantaneous rate of change is exactly this average rate.

Theorem 10.7 (Mean Value Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Subtract from f the secant line joining its endpoint values. More precisely, define

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function g is continuous on $[a, b]$, differentiable on (a, b) , and $g(a) = g(b) = 0$. Rolle's theorem gives a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

the desired equality follows. □

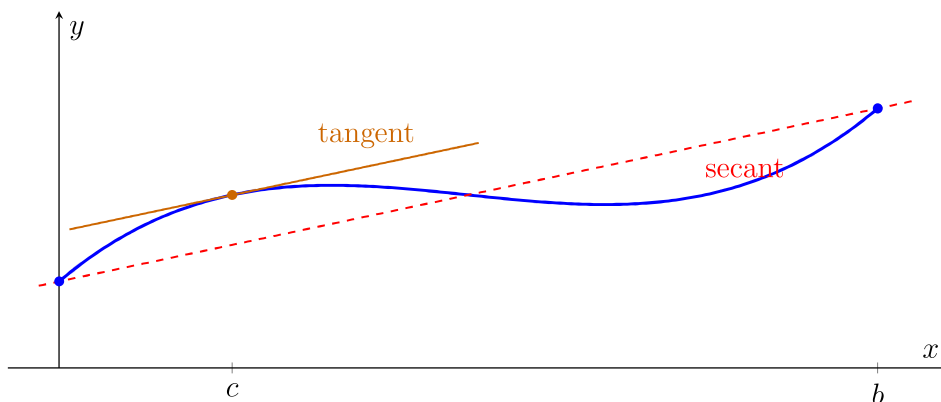


Figure 29: At the point c , the tangent is parallel to the secant through the endpoints.

Example 10.8. Apply the Mean Value Theorem to $f(x) = \sqrt{x}$ on $[1, 4]$. The function is continuous on $[1, 4]$ and differentiable on $(1, 4)$. Its average rate of change is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$

We therefore look for $c \in (1, 4)$ such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

It follows that $2\sqrt{c} = 3$, so

$$c = \frac{9}{4}.$$

Notice that the theorem first guarantees existence. Solving the equation then identifies the point in this particular example.

10.4 Consequences of the Mean Value Theorem

The first consequence is the precise relation between the sign of the derivative and monotonicity.

Proposition 10.9 (Derivative and monotonicity). Let f be continuous on an interval I and differentiable in its interior.

- If $f'(x) \geq 0$ for every interior point x , then f is increasing on I .
- If $f'(x) > 0$ for every interior point x , then f is strictly increasing on I .
- If $f'(x) \leq 0$ for every interior point x , then f is decreasing on I .
- If $f'(x) < 0$ for every interior point x , then f is strictly decreasing on I .

Indeed, if $x < y$ are points of I , the Mean Value Theorem applied on $[x, y]$ gives

$$f(y) - f(x) = f'(c)(y - x)$$

for some $c \in (x, y)$. Since $y - x > 0$, the sign of the difference $f(y) - f(x)$ is controlled by the sign of $f'(c)$.

Corollary 10.10. If $f'(x) = 0$ at every interior point of an interval I , then f is constant on I .

This statement depends on the fact that the domain is an interval. A function may have derivative zero on a disconnected domain and take different constant values on different components.

The Mean Value Theorem also turns a bound on the derivative into a bound on the change of the function.

Proposition 10.11 (A useful estimate). Suppose that f is continuous on $[a, b]$, differentiable on (a, b) , and

$$|f'(x)| \leq M$$

for every $x \in (a, b)$. Then

$$|f(b) - f(a)| \leq M|b - a|.$$

Example 10.12. Since $|\cos x| \leq 1$, the Mean Value Theorem applied to the sine function gives

$$|\sin y - \sin x| \leq |y - x|$$

for all $x, y \in \mathbb{R}$. In particular, taking $y = 0$, we obtain the familiar inequality

$$|\sin x| \leq |x|.$$

This argument explains why the estimate is true and not only how to remember it.

Example 10.13. Let

$$p(x) = x^3 + ax^2 + bx + c,$$

where $a^2 < 3b$. We claim that $p(x) = 0$ has exactly one real solution. First, because $p(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $p(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, continuity gives at least one zero. Moreover,

$$p'(x) = 3x^2 + 2ax + b.$$

The discriminant of this quadratic is

$$4a^2 - 12b < 0.$$

Since its leading coefficient is positive, $p'(x) > 0$ for every $x \in \mathbb{R}$. Thus, p is strictly increasing and can have at most one zero. The zero is therefore unique.

10.5 Existence and uniqueness of solutions

Calculus often separates a problem into two logically different questions. Continuity is well suited to proving that a solution exists, while monotonicity is well suited to proving that it is unique. It is useful to say explicitly which theorem is doing which job.

Example 10.14. Consider the equation

$$\cos x = x.$$

Define $F(x) = \cos x - x$. Since

$$F(0) = 1 > 0, \quad F(1) = \cos 1 - 1 < 0,$$

the Intermediate Value Theorem gives a solution in $(0, 1)$. This proves existence. On the other hand,

$$F'(x) = -\sin x - 1 < 0$$

for $0 < x < 1$. Hence F is strictly decreasing on $[0, 1]$ and can vanish at most once there. In fact, there cannot be a solution outside $[-1, 1]$ because $|\cos x| \leq 1$; and on $[-1, 0]$ we have $\cos x > 0 \geq x$. Thus the real solution is unique.

The numerical value is approximately 0.739, but neither the Intermediate Value Theorem nor the Mean Value Theorem is intended to calculate that value. Numerical methods will address such approximations in another course.

The same combination of ideas can control how many times a graph intersects a line. To study $f(x) = mx + b$, define $H(x) = f(x) - mx - b$. If H' never vanishes, then H is strictly monotone and the intersection is unique whenever it exists. If H' has at most k zeros, Rolle's theorem shows that H has at most $k + 1$ zeros.

Example 10.15. Suppose that f is differentiable on $\mathbb{R}$ and

$$2 \leq f'(x) \leq 5$$

for every $x \in \mathbb{R}$. Applying the Mean Value Theorem between x and y , with $x < y$, gives

$$2(y - x) \leq f(y) - f(x) \leq 5(y - x).$$

The lower bound shows that f is strictly increasing. It also shows that the function cannot flatten out: increasing the input by one unit increases the output by at least two units. The upper bound controls how rapidly the output can change. Such two-sided estimates occur frequently in stability and error analysis.

10.6 Checking the hypotheses

The three main theorems of this lecture have similar hypotheses, and it is easy to quote one of them too quickly. A reliable solution should check:

- continuity on the whole closed interval, including both endpoints;
- differentiability only on the open interval;

- equality of endpoint values when Rolle's theorem is used;
- that the point eventually obtained belongs to the open interval.

For example, $f(x) = 1/x$ satisfies $f(-1) = f(1)$, but Rolle's theorem does not apply on $[-1, 1]$ because the function is not even defined, much less continuous, at zero. Indeed, $f'(x) = -1/x^2$ never vanishes. Similarly, the point obtained by solving the derivative equation may be an endpoint; such a point does not satisfy the conclusion of the theorem. Checking membership in (a, b) is part of the solution, not an optional detail.

Remark 10.16. The Mean Value Theorem is an existence theorem. It does not say that the point c is unique, and it does not give a general algorithm for finding it. Its strength lies in the consequences that follow even when c is unknown.