

11 Lecture #8: Week 4

11.1 The Cauchy Mean Value Theorem

The ordinary Mean Value Theorem compares the change in a function with the change in the identity function $x \mapsto x$. A more symmetric version compares the changes of two functions. This is the form that lies behind L'Hospital's rule.

Theorem 11.1 (Cauchy Mean Value Theorem). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

If $g(b) \neq g(a)$ and $g'(c) \neq 0$, this can be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Consider

$$h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a)).$$

We have $h(a) = h(b) = 0$. Rolle's theorem gives $c \in (a, b)$ with $h'(c) = 0$, and this equality is exactly the conclusion of the theorem. $\square$

Taking $g(x) = x$ recovers the ordinary Mean Value Theorem.

11.2 L'Hospital's rule for the form 0/0

Suppose that both numerator and denominator tend to zero. The quotient is then not determined by the two limits alone. L'Hospital's rule tells us that, under appropriate hypotheses, the relative speeds at which the two functions approach zero can be measured by the quotient of their derivatives.

Theorem 11.2 (L'Hospital's rule: the form 0/0). Let f and g be differentiable on a deleted neighbourhood of a , with

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0.$$

Assume that $g'(x) \neq 0$ near a , except possibly at a , and that the limit

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$$

exists, where L may be finite or infinite. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

The same statement holds for one-sided limits.

To see the main idea when the functions extend continuously by $f(a) = g(a) = 0$, take x close to a and apply the Cauchy Mean Value Theorem on the interval with endpoints a and x . For some point c_x between them,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c_x)}{g'(c_x)}.$$

As $x \rightarrow a$, the point c_x also tends to a , which explains the conclusion.

Example 11.3. Let us calculate

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$$

The quotient has the form $0/0$. Applying L'Hospital's rule once gives

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{2x},$$

which is still of the form $0/0$. A second application gives

$$\lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

Each application is justified because the quotient at that stage is indeterminate and the relevant derivatives exist near the point.

Example 11.4. For $x > -1$,

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1.$$

This limit is also the derivative of $\ln(1+x)$ at the origin. L'Hospital's rule is consistent with that interpretation.

11.3 The form ∞/∞

There is a second basic version of the rule.

Theorem 11.5 (L'Hospital's rule: the form ∞/∞). Suppose that f and g are differentiable near a , that $g'(x) \neq 0$, and that

$$|f(x)| \rightarrow \infty, \quad |g(x)| \rightarrow \infty$$

as $x \rightarrow a$. If

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

The point a may also be $+\infty$ or $-\infty$.

Example 11.6. The logarithm grows more slowly than every positive power. For the simplest case,

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \lim_{x \rightarrow +\infty} \frac{1/x}{1} = 0.$$

Similarly, the exponential grows faster than every polynomial. For instance,

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^x} = \lim_{x \rightarrow +\infty} \frac{2x}{e^x} = \lim_{x \rightarrow +\infty} \frac{2}{e^x} = 0.$$

The same idea gives a useful hierarchy of growth. If $\alpha > 0$ and $m \in \mathbb{N}$, then repeated applications of L'Hospital's rule yield

$$\lim_{x \rightarrow +\infty} \frac{(\ln x)^m}{x^\alpha} = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{x^m}{e^{\alpha x}} = 0.$$

One should not read this as saying that L'Hospital's rule is always the shortest way to compare functions. Rather, it confirms the qualitative order

$$\text{logarithms} \ll \text{powers} \ll \text{exponentials}$$

as $x \rightarrow +\infty$.

Example 11.7. Let $m \in \mathbb{N}$. We claim that

$$\lim_{x \rightarrow +\infty} \frac{x^m}{e^x} = 0.$$

After one application of L'Hospital's rule, the power drops from m to $m - 1$. After m applications, the quotient becomes

$$\frac{m!}{e^x},$$

whose limit is zero. This calculation is legitimate because at every preceding stage the quotient still has the form ∞/∞ . It also explains why an exponential eventually dominates any fixed polynomial, even a polynomial of very high degree.

11.4 When the rule should not be used

L'Hospital's rule is not an instruction to differentiate every quotient. Before using it, verify that the quotient has the form $0/0$ or ∞/∞ .

For example,

$$\lim_{x \rightarrow 0} \frac{x^2 + 1}{x + 1} = 1.$$

If one differentiated numerator and denominator, one would obtain the incorrect answer 0. The original quotient is not indeterminate, so the rule does not apply.

The converse of L'Hospital's rule is also false: the original quotient may have a limit even when the quotient of derivatives does not. Indeed,

$$\lim_{x \rightarrow +\infty} \frac{x + \sin x}{x} = 1,$$

but

$$\frac{1 + \cos x}{1} = 1 + \cos x$$

has no limit. In this example L'Hospital's rule is simply inconclusive; dividing the original quotient by x solves the problem immediately.

11.5 Products and differences

The forms $0 \cdot \infty$ and $\infty - \infty$ are not direct L'Hospital forms. They must first be rewritten.

Example 11.8. Consider

$$\lim_{x \rightarrow 0^+} x \ln x.$$

This has the form $0 \cdot (-\infty)$. We rewrite it as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

Now the quotient has the form $-\infty/\infty$, and

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

Thus $x \ln x \rightarrow 0$ as $x \rightarrow 0^+$.

Example 11.9. Let us calculate

$$\lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - x).$$

The form is $\infty - \infty$. Rationalizing is more efficient than differentiating:

$$\begin{aligned} \sqrt{x^2 + x} - x &= \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} \\ &= \frac{x}{\sqrt{x^2 + x} + x} = \frac{1}{\sqrt{1 + 1/x} + 1}. \end{aligned}$$

Therefore, the limit is $1/2$.

11.6 Indeterminate powers

The standard indeterminate powers are

$$0^0, \quad 1^\infty, \quad \infty^0.$$

The systematic method is to take logarithms. If

$$y = f(x)^{g(x)},$$

with $f(x) > 0$ near the point, then

$$\ln y = g(x) \ln f(x).$$

We first calculate the limit of the logarithm and then exponentiate.

Example 11.10. For $x > 0$,

$$y = (1 + x)^{1/x}$$

has the indeterminate form 1^∞ as $x \rightarrow 0^+$. Taking logarithms,

$$\ln y = \frac{\ln(1 + x)}{x}.$$

The limit on the right is 1, so

$$\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = e.$$

Example 11.11. Consider, for $x > 0$ close to zero,

$$y = (\sin x + \cos x)^{1/x}.$$

The base tends to 1 and the exponent tends to $+\infty$. We take logarithms:

$$\ln y = \frac{\ln(\sin x + \cos x)}{x}.$$

This is a $0/0$ quotient. L'Hospital's rule gives

$$\lim_{x \rightarrow 0^+} \frac{\cos x - \sin x}{\sin x + \cos x} = 1.$$

Consequently,

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x} = e.$$

Algorithm 11.12 (A checklist for indeterminate limits). (1) Substitute first and identify the form.

(2) If the form is $0/0$ or ∞/∞ , check the hypotheses before using L'Hospital's rule.

(3) Rewrite $0 \cdot \infty$ as a quotient.

(4) Combine or rationalize an expression of the form $\infty - \infty$.

(5) For an indeterminate power, take logarithms, solve the new limit, and exponentiate.

(6) After differentiating, check the new limit again. Do not continue automatically if the new quotient is no longer indeterminate.