

Calculus 1

Lecture 7: Mean Value Theorems

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What we will be able to do today

- Identify local extrema and critical points.
- Apply Fermat's and Rolle's theorems with all hypotheses checked.
- Interpret the Mean Value Theorem geometrically and analytically.
- Use derivatives to prove monotonicity, estimates and uniqueness.
- Separate the existence of a solution from its uniqueness.

Local extrema and critical points

Local means near the point

Let f be defined on an interval I and let $c \in I$.

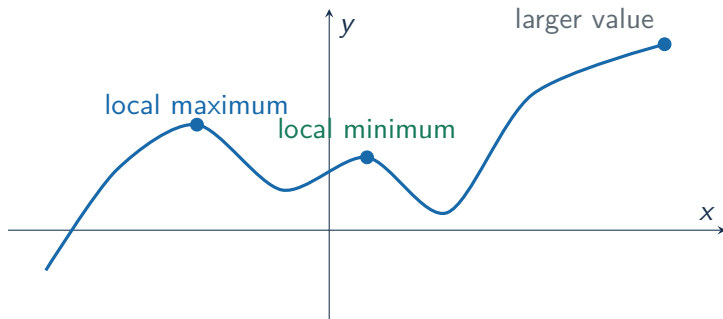
f has a **local maximum** at c if, for some $\delta > 0$,

$$f(x) \leq f(c) \quad \text{whenever } x \in I \text{ and } |x - c| < \delta.$$

f has a **local minimum** at c if

$$f(c) \leq f(x) \quad \text{whenever } x \in I \text{ and } |x - c| < \delta.$$

A local maximum need not be the largest value



We compare only with values in a sufficiently small neighbourhood.

Fermat's theorem gives a necessary condition

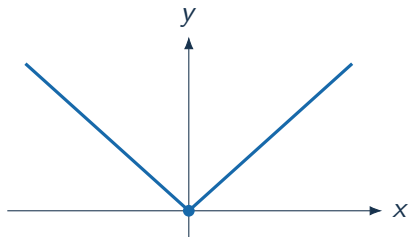
Let c be an interior point of an interval I . If $f : I \rightarrow \mathbb{R}$ is differentiable at c and has a local extremum there, then

$$f'(c) = 0.$$

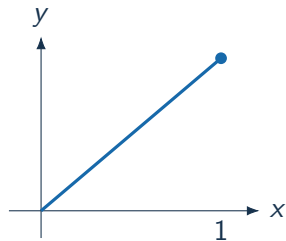
The hypotheses matter:

- c must be an interior point;
- f must be differentiable at c .

A corner and an endpoint show why the hypotheses matter



$f(x) = |x|$ has a minimum at 0, but $f'(0)$ does not exist.



$f(x) = x$ on $[0, 1]$ has its maximum at the endpoint, but the slope is 1.

Critical points are candidates, not conclusions

An interior point c is a **critical point** of f if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Fermat's theorem says:

$$\text{differentiable local extremum} \implies \text{critical point.}$$

The converse is false: $f(x) = x^3$ has $f'(0) = 0$, but no extremum at 0.

Rolle's theorem

Rolle's theorem: equal endpoint heights

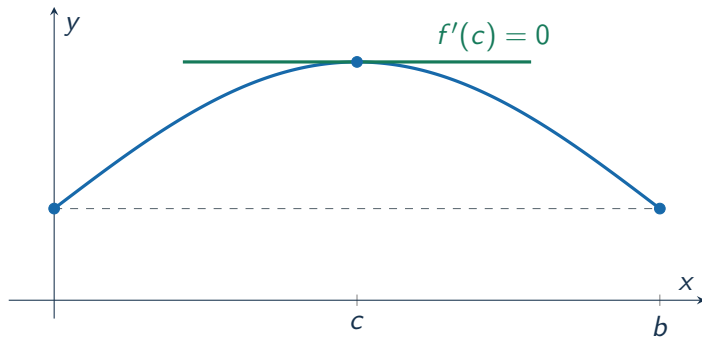
Let $f : [a, b] \rightarrow \mathbb{R}$ be

- continuous on $[a, b]$,
- differentiable on (a, b) ,
- and satisfy $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Rolle's theorem guarantees a horizontal tangent



The theorem guarantees at least one such point, not exactly one.

Why Rolle's theorem is true

By the Extreme Value Theorem, f attains a maximum and a minimum on $[a, b]$.

- If the maximum equals the minimum, f is constant and $f'(c) = 0$ everywhere.
- Otherwise, at least one extremum is attained inside (a, b) because the two endpoint values are equal.
- Fermat's theorem gives $f'(c) = 0$ at that interior extremum.

Example: apply Rolle's theorem to $x^3 - x$

Let

$$f(x) = x^3 - x \quad \text{on } [-1, 1].$$

The function is continuous on $[-1, 1]$, differentiable on $(-1, 1)$, and

$$f(-1) = 0 = f(1).$$

Therefore Rolle's theorem applies.

Example: locate all Rolle points

Differentiate and solve:

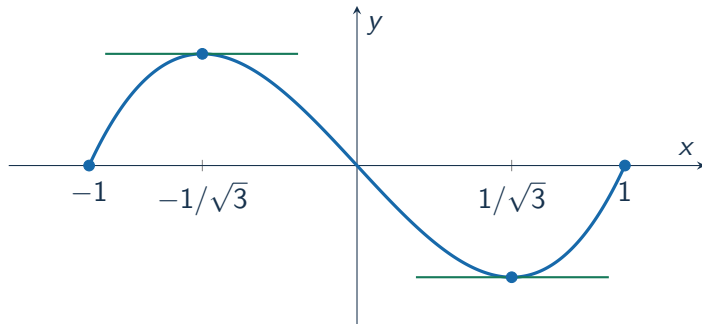
$$f'(x) = 3x^2 - 1,$$

$$3c^2 - 1 = 0 \quad \Longleftrightarrow \quad c = \pm \frac{1}{\sqrt{3}}.$$

Both values lie in $(-1, 1)$.

$$c = -1/\sqrt{3} \text{ and } c = 1/\sqrt{3}$$

The example has two horizontal tangents



Rolle's theorem can count zeros

★ If a differentiable function has two distinct zeros, then its derivative has a zero between them.

The Mean Value Theorem

Average change becomes instantaneous change

The average rate of change of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

It is the slope of the secant through the endpoint values.

The Mean Value Theorem finds an interior point where the tangent has exactly this slope.

The Mean Value Theorem

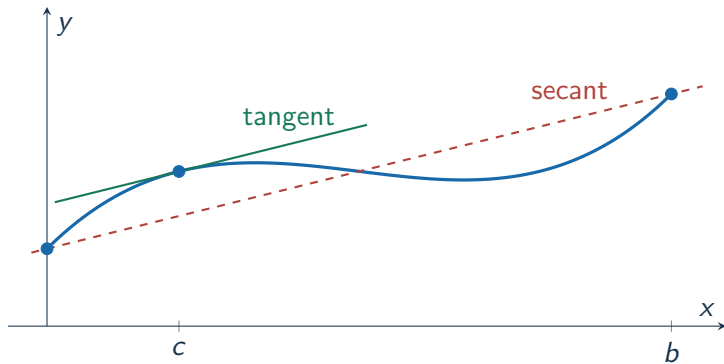
Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) .
Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

At c , the tangent is parallel to the secant



Example: check the hypotheses for $\sqrt{x}$

Apply the theorem to

$$f(x) = \sqrt{x} \quad \text{on } [1, 4].$$

- $\sqrt{x}$ is continuous on $[1, 4]$.
- It is differentiable on $(1, 4)$.

Its average rate of change is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$

Example: solve the derivative equation

We need $c \in (1, 4)$ such that

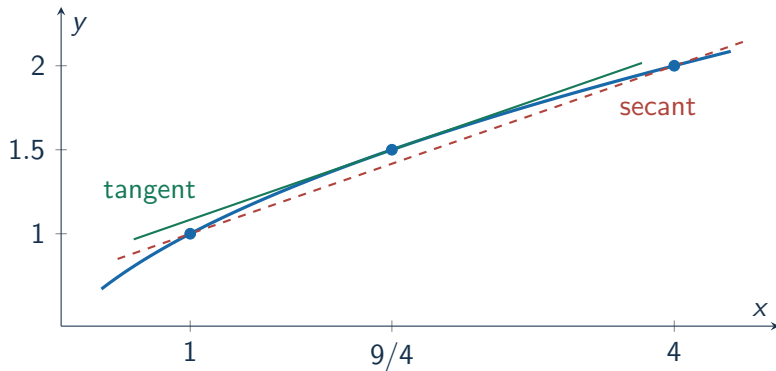
$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Therefore,

$$2\sqrt{c} = 3 \implies c = \frac{9}{4}.$$

$$c = 9/4$$

The tangent and secant really are parallel



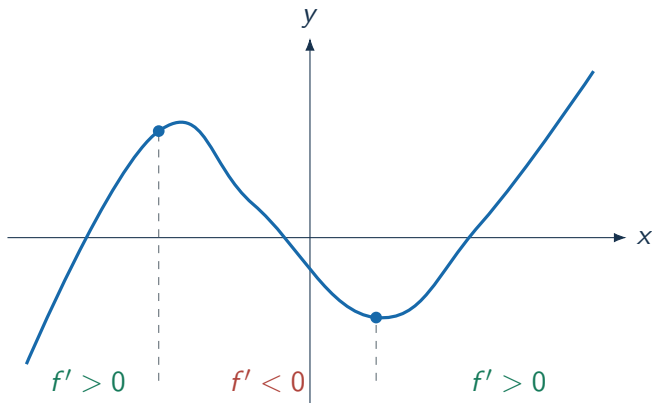
Consequences and applications

The sign of the derivative controls monotonicity

Let f be continuous on an interval and differentiable in its interior.

$f'(x) > 0$	f is strictly increasing,
$f'(x) < 0$	f is strictly decreasing,
$f'(x) = 0$	f is constant.

A derivative sign chart predicts the graph



A derivative bound controls total change

Suppose that f is continuous on $[a, b]$, differentiable on (a, b) , and

$$|f'(x)| \leq M.$$

The Mean Value Theorem gives

$$f(b) - f(a) = f'(c)(b - a).$$

Taking absolute values,

$$|f(b) - f(a)| \leq M |b - a|.$$

Example: recover $|\sin x| \leq |x|$

Since

$$|(\sin x)'| = |\cos x| \leq 1,$$

the derivative estimate gives

$$|\sin y - \sin x| \leq |y - x|.$$

Taking $y = 0$,

$$|\sin x - \sin 0| \leq |x - 0|.$$

$$|\sin x| \leq |x| \text{ for every } x \in \mathbb{R}$$

Existence and uniqueness use different ideas

To study an equation $F(x) = 0$:

Existence

Continuity and a sign change give a zero by the Intermediate Value Theorem.

Uniqueness

A derivative with one strict sign makes F strictly monotone, so it has at most one zero.

Example: prove that $\cos x = x$ has a solution

Define

$$F(x) = \cos x - x.$$

Then

$$F(0) = 1 > 0, \quad F(1) = \cos 1 - 1 < 0.$$

Because F is continuous, the Intermediate Value Theorem gives a solution in $(0, 1)$.

Example: prove that the solution is unique

On $(0, 1)$,

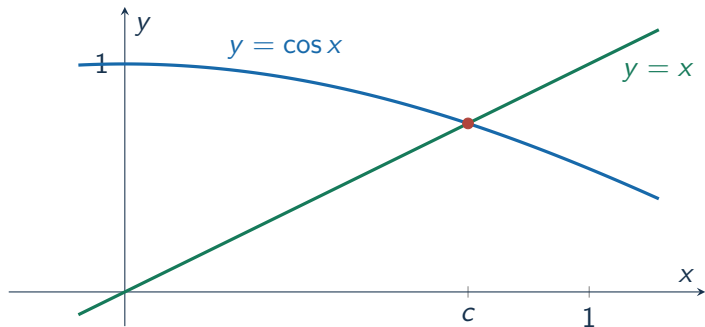
$$F'(x) = -\sin x - 1 < 0.$$

Thus F is strictly decreasing and can vanish at most once on $[0, 1]$.

There are no solutions outside $[-1, 1]$ because $|\cos x| \leq 1$, and none in $[-1, 0]$ because $\cos x > 0 \geq x$.

The equation $\cos x = x$ has exactly one real solution.

The unique intersection lies between 0 and 1



A reliable theorem checklist

Check the hypotheses before solving for c

1. Is the function continuous on the entire closed interval $[a, b]$?
2. Is it differentiable on the open interval (a, b) ?
3. For Rolle's theorem, do we have $f(a) = f(b)$?
4. After solving the derivative equation, does the candidate lie in (a, b) ?

A tempting example where Rolle's theorem fails

For

$$f(x) = \frac{1}{x} \quad \text{on } [-1, 1],$$

we have

$$f(-1) = -1, \quad f(1) = 1.$$

More importantly, the function is not defined at 0, so it is not continuous on $[-1, 1]$.
No conclusion from Rolle's theorem is available.