

Calculus 1

Lecture 8: L'Hospital's Rule and Indeterminate Forms

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What we will be able to do today

- Compare the changes of two functions.
- Apply L'Hospital's rule to the forms $0/0$ and ∞/∞ .
- Recognize situations in which the rule is not applicable.
- Rewrite products, differences and indeterminate powers.
- Compare the growth of logarithms, powers and exponentials.

The form $0/0$

An indeterminate form is a diagnosis, not an answer

If

$$f(x) \rightarrow 0, \quad g(x) \rightarrow 0,$$

then the quotient $f(x)/g(x)$ may approach a finite number, an infinity, or no limit at all.

The symbol

$$\frac{0}{0}$$

only tells us that more analysis is needed.

L'Hospital's rule for $0/0$

Suppose that, near a ,

- f and g are differentiable,
- $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$,
- $g'(x) \neq 0$,
- and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ exists.

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

The same statement holds for one-sided limits.

Example: identify the form before differentiating

Calculate

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$$

Substitution gives

$$\frac{e^0 - 1 - 0}{0^2} = \frac{0}{0}.$$

Therefore, the first application of L'Hospital's rule is permitted.

Example: the first application is not enough

Differentiate numerator and denominator:

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x}.$$

The new quotient is still 0/0, so the rule may be applied again:

$$\lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

$$1/2$$

Example: a standard logarithmic limit

For $x > -1$,

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$

has the form $0/0$.

Hence,

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1.$$

This is also the derivative of $\ln(1+x)$ at $x = 0$.

The form ∞/∞

The second basic version of the rule

Suppose that f and g are differentiable near a , $g'(x) \neq 0$, and

$$|f(x)| \rightarrow \infty, \quad |g(x)| \rightarrow \infty.$$

If

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.}$$

The point a may also be $+\infty$ or $-\infty$.

Example: logarithms grow more slowly than powers

Consider

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x}.$$

This is an ∞/∞ form. Therefore,

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \lim_{x \rightarrow +\infty} \frac{1/x}{1} = 0.$$

$$\ln x \ll x \text{ as } x \rightarrow +\infty$$

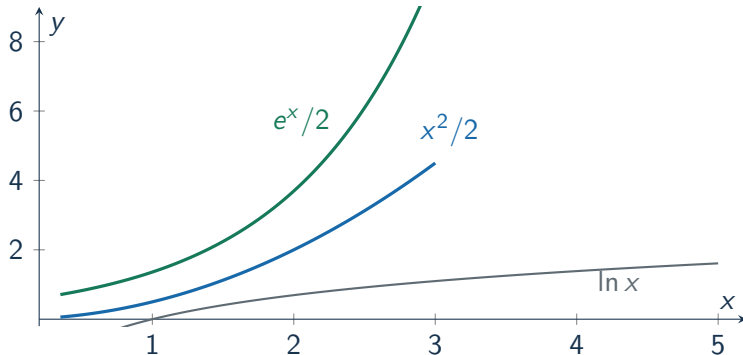
Example: exponentials grow faster than polynomials

For instance,

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^x} = \lim_{x \rightarrow +\infty} \frac{2x}{e^x} = \lim_{x \rightarrow +\infty} \frac{2}{e^x} = 0.$$

Each application is justified because the previous quotient still has the form ∞/∞ .

The growth hierarchy becomes visible



logarithms $\ll$ powers $\ll$ exponentials.

When the rule does not apply

When can we use L'Hospital's rule?

L'Hospital's rule applies directly only to a quotient with one of the indeterminate forms

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Before differentiating, we must check:

- that the expression is a quotient;
- that the quotient has the correct indeterminate form;
- that the differentiability hypotheses hold;
- that the quotient of the derivatives has a limit.

Incorrect use 1: the quotient is not indeterminate

Consider

$$\lim_{x \rightarrow 0} \frac{1 + x^2}{1 + x}.$$

A student might incorrectly write

$$\lim_{x \rightarrow 0} \frac{1 + x^2}{1 + x} = \lim_{x \rightarrow 0} \frac{2x}{1} = 0.$$

However, the original quotient has the form

$$\frac{1}{1},$$

not $0/0$ or ∞/∞ .

Therefore, L'Hospital's rule cannot be used.

Incorrect use 1: the correct solution

The denominator is nonzero at $x = 0$, so we may use direct substitution:

$$\lim_{x \rightarrow 0} \frac{1 + x^2}{1 + x} = \frac{1 + 0^2}{1 + 0} = 1.$$

Thus,

$$\boxed{\lim_{x \rightarrow 0} \frac{1 + x^2}{1 + x} = 1}.$$

Lesson

Always identify the indeterminate form before differentiating.

Incorrect use 2: an infinity minus infinity

Consider

$$\lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 4x} - x \right).$$

A student might differentiate the two terms and write

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 4x} - x \right) &= \lim_{x \rightarrow +\infty} \left(\frac{x+2}{\sqrt{x^2 + 4x}} - 1 \right) \\ &= 1 - 1 = 0. \end{aligned}$$

This is incorrect: the expression is a difference, not a quotient.

Incorrect use 2: the correct solution

The expression has the indeterminate form

$$\infty - \infty.$$

We first rationalize:

$$\begin{aligned}\sqrt{x^2 + 4x} - x &= \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x} \\ &= \frac{4x}{\sqrt{x^2 + 4x} + x} \\ &= \frac{4}{\sqrt{1 + 4/x} + 1}.\end{aligned}$$

Therefore,

$$\boxed{\lim_{x \rightarrow +\infty} (\sqrt{x^2 + 4x} - x) = 2}.$$

Incorrect use 3: the derivative quotient oscillates

Consider

$$\lim_{x \rightarrow +\infty} \frac{x + \sin x}{x}.$$

This is an ∞/∞ form. Differentiating gives

$$\frac{(x + \sin x)'}{x'} = 1 + \cos x.$$

The function $1 + \cos x$ has no limit as $x \rightarrow +\infty$.

A student might then conclude that

$$\lim_{x \rightarrow +\infty} \frac{x + \sin x}{x}$$

does not exist. This conclusion is incorrect.

Incorrect use 3: the correct solution

Rewrite the original quotient:

$$\frac{x + \sin x}{x} = 1 + \frac{\sin x}{x}.$$

Since $-1 \leq \sin x \leq 1$, for $x > 0$ we have

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$$

By the squeeze theorem,

$$\frac{\sin x}{x} \longrightarrow 0.$$

Consequently,

$$\boxed{\lim_{x \rightarrow +\infty} \frac{x + \sin x}{x} = 1.}$$

What did the third example teach us?

L'Hospital's rule says that, under the appropriate hypotheses,

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \quad \implies \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

It does not say that

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \text{ does not exist}$$

implies that the original limit does not exist.

Three common mistakes

1. Applying L'Hospital's rule to a quotient that is not of the form $0/0$ or ∞/∞ .
2. Differentiating the terms of a sum or difference.
3. Concluding that the original limit does not exist because the quotient of the derivatives has no limit.

A short decision rule

Before applying L'Hospital's rule:

1. Substitute or analyse the numerator and denominator separately.
2. Verify that the form is $0/0$ or ∞/∞ .
3. Differentiate the numerator and denominator separately.
4. Check the new limit again before repeating the rule.

Products and differences

A product $0 \cdot \infty$ must be rewritten

Calculate

$$\lim_{x \rightarrow 0^+} x \ln x.$$

The direct form is $0 \cdot (-\infty)$, so L'Hospital's rule does not yet apply.

Rewrite

$$x \ln x = \frac{\ln x}{1/x}.$$

Now the quotient has the form $-\infty/\infty$.

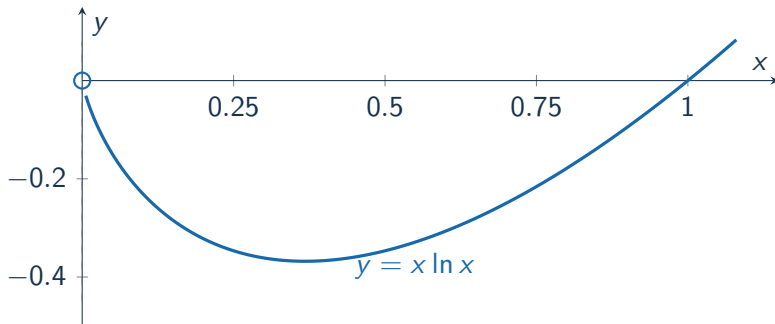
Example: finish the product limit

Apply the rule:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

$$\lim_{x \rightarrow 0^+} x \ln x = 0$$

The logarithm diverges, but the product still tends to zero



A difference $\infty - \infty$ should be combined

Calculate

$$\lim_{x \rightarrow +\infty} (\sqrt{x^2 + x} - x).$$

Rationalization gives

$$\begin{aligned}\sqrt{x^2 + x} - x &= \frac{x}{\sqrt{x^2 + x} + x} \\ &= \frac{1}{\sqrt{1 + 1/x} + 1}.\end{aligned}$$

Therefore, the limit is $1/2$.

Indeterminate powers

Three common indeterminate powers

The standard forms are

$$0^0, \quad 1^\infty, \quad \infty^0.$$

For

$$y = f(x)^{g(x)}, \quad f(x) > 0,$$

take logarithms:

$$\ln y = g(x) \ln f(x).$$

First solve the logarithmic limit, then exponentiate.

Example: the basic 1^∞ limit

Let

$$y = (1 + x)^{1/x} \quad (x > 0).$$

Taking logarithms,

$$\ln y = \frac{\ln(1 + x)}{x}.$$

The right-hand side tends to 1. Hence,

$$y \rightarrow e^1 = e.$$

$$\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = e$$

Example: a trigonometric base

For $x > 0$ near zero, put

$$y = (\sin x + \cos x)^{1/x}.$$

Then

$$\ln y = \frac{\ln(\sin x + \cos x)}{x}.$$

This quotient has the form $0/0$.

Example: finish with L'Hospital and exponentiation

Applying the rule,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\cos x - \sin x}{\sin x + \cos x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x} = e.$$

e

A checklist for indeterminate limits

Choose the transformation from the form

$0/0, \infty/\infty$

Check the hypotheses, then consider L'Hospital.

$0 \cdot \infty$

Rewrite as a quotient.

$\infty - \infty$

Combine fractions or rationalize.

$0^0, 1^\infty, \infty^0$

Take logarithms, solve, then exponentiate.

Is this solution correct?

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x}{e^x - 1}.$$

Since the quotient has the form $0/0$, we apply L'Hospital's rule:

$$\lim_{x \rightarrow 0} \frac{\sin x}{e^x - 1} = \lim_{x \rightarrow 0} \frac{\cos x}{e^x} = \lim_{x \rightarrow 0} \frac{-\sin x}{e^x} = 0.$$

Is this solution correct?

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Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x}{e^x - 1}.$$

Since the quotient has the form $0/0$, we apply L'Hospital's rule:

$$\lim_{x \rightarrow 0} \frac{\sin x}{e^x - 1} = \lim_{x \rightarrow 0} \frac{\cos x}{e^x} = \lim_{x \rightarrow 0} \frac{-\sin x}{e^x} = 0.$$

Is this solution correct?

The hidden mistake

After the first application, the quotient has the form

$$\frac{\cos 0}{e^0} = \frac{1}{1}.$$