

Calculus 1

Practice 4: Mean Value Theorems and Indeterminate Limits

Sheldon Dantas

Faculty of Electrical Engineering, Czech Technical University in Prague

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Seven problems review the whole week

1. Rolle's and Mean Value Theorems
2. Existence and uniqueness of a cubic zero
3. Inequalities from monotonicity
4. Direct applications of L'Hospital's rule
5. Products and differences
6. Indeterminate powers
7. Exact counting of real solutions

Problem 1: Rolle and Mean Value points

Problem 1

- (a) Find all Rolle points for $f(x) = x^3 - x$ on $[-1, 1]$.
- (b) Explain why Rolle's theorem does not apply to $g(x) = |x|$ on $[-1, 1]$, and verify that its conclusion is false.
- (c) Find the Mean Value point for $h(x) = x^3 - 2x$ on $[-1, 2]$.

Problem 1(a): verify the theorem before solving

$f(x) = x^3 - x$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

Moreover,

$$f(-1) = 0 = f(1).$$

Thus Rolle's theorem applies. Solve

$$f'(c) = 3c^2 - 1 = 0.$$

$$c = \pm 1/\sqrt{3}$$

Problem 1(b): one missing hypothesis destroys the conclusion

$g(x) = |x|$ is continuous and satisfies $g(-1) = g(1) = 1$.

However, it is not differentiable at 0.

For every point where the derivative exists,

$$g'(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

There is no $c \in (-1, 1)$ with $g'(c) = 0$.

Problem 1(c): calculate the average slope first

For $h(x) = x^3 - 2x$ on $[-1, 2]$,

$$\frac{h(2) - h(-1)}{2 - (-1)} = \frac{4 - 1}{3} = 1.$$

Since $h'(x) = 3x^2 - 2$, solve

$$3c^2 - 2 = 1 \iff c^2 = 1.$$

The algebraic solutions are -1 and 1 , but only 1 belongs to $(-1, 2)$.

$$c = 1$$

Problem 2: exactly one real zero

Problem 2

Let

$$p(x) = x^3 + ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$ and $a^2 < 3b$.

Prove that $p(x) = 0$ has exactly one real solution.

We will prove existence and uniqueness separately.

Problem 2: continuity gives existence

Because the leading term is x^3 ,

$$\lim_{x \rightarrow -\infty} p(x) = -\infty, \quad \lim_{x \rightarrow +\infty} p(x) = +\infty.$$

The polynomial is continuous. Hence, for sufficiently large R ,

$$p(-R) < 0 < p(R).$$

The Intermediate Value Theorem gives at least one real zero.

Problem 2: the derivative gives uniqueness

Differentiate:

$$p'(x) = 3x^2 + 2ax + b.$$

Its discriminant is

$$\Delta = (2a)^2 - 4 \cdot 3b = 4a^2 - 12b < 0.$$

Since the leading coefficient is positive,

$$p'(x) > 0 \quad \text{for every } x \in \mathbb{R}.$$

Thus p is strictly increasing and has at most one zero.

$$p(x) = 0 \text{ has exactly one real solution.}$$

Problem 3: two fundamental inequalities

Problem 3

Prove

$$e^x \geq 1 + x \quad (x \in \mathbb{R})$$

and

$$\ln x \leq x - 1 \quad (x > 0).$$

Determine when equality holds.

Problem 3(a): convert the inequality into a minimum problem

Define

$$F(x) = e^x - x - 1.$$

Then

$$F'(x) = e^x - 1.$$

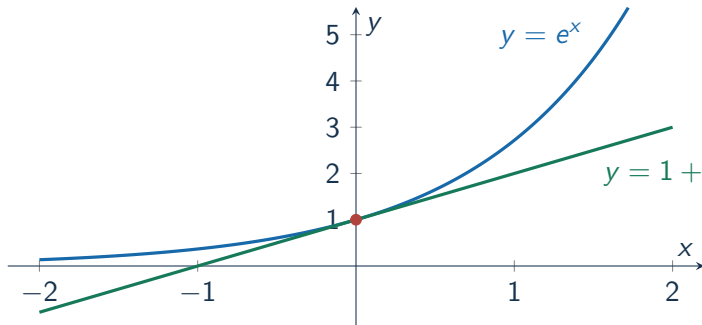
Thus $F' < 0$ for $x < 0$, $F'(0) = 0$, and $F' > 0$ for $x > 0$.

Therefore, F has an absolute minimum at 0:

$$F(0) = 0.$$

$$e^x \geq 1 + x, \text{ with equality only at } x = 0.$$

The tangent line lies below the exponential



Problem 3(b): repeat the monotonicity argument

Define

$$G(x) = x - 1 - \ln x, \quad x > 0.$$

Then

$$G'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}.$$

G decreases on $(0, 1)$ and increases on $(1, +\infty)$. Since $G(1) = 0$,

$$G(x) \geq 0.$$

$$\ln x \leq x - 1, \text{ with equality only at } x = 1.$$

Problem 4: direct L'Hospital limits

Problem 4

Calculate using L'Hospital's rule:

$$(a) \quad \lim_{x \rightarrow 0} \frac{e^{2x} - 1 - 2x}{x^2},$$

$$(b) \quad \lim_{x \rightarrow +\infty} \frac{\ln x}{\sqrt{x}},$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

Problem 4(a): two justified applications

The original form is 0/0:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{2x} - 1 - 2x}{x^2} &= \lim_{x \rightarrow 0} \frac{2e^{2x} - 2}{2x} \\ &= \lim_{x \rightarrow 0} \frac{4e^{2x}}{2} = 2.\end{aligned}$$

After the first differentiation, the form is still 0/0.

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Problem 4(b): logarithm versus square root

The form is ∞/∞ :

$$\begin{aligned}\lim_{x \rightarrow +\infty} \frac{\ln x}{\sqrt{x}} &= \lim_{x \rightarrow +\infty} \frac{1/x}{1/(2\sqrt{x})} \\ &= \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{x}} = 0.\end{aligned}$$

0

Problem 4(c): differentiate until the form is resolved

Three successive 0/0 applications give

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{-\sin x}{6x} \\ &= \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}.\end{aligned}$$

$$-1/6$$

Problem 5: rewrite before applying a rule

Problem 5

Calculate:

$$(a) \quad \lim_{x \rightarrow 0^+} x \ln x,$$

$$(b) \quad \lim_{x \rightarrow +\infty} (\sqrt{x^2 + 3x} - x),$$

$$(c) \quad \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right).$$

Problem 5(a): turn the product into a quotient

Rewrite

$$x \ln x = \frac{\ln x}{1/x}.$$

Now apply L'Hospital:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

0

Problem 5(b): rationalization is the efficient move

$$\begin{aligned}\sqrt{x^2 + 3x} - x &= \frac{3x}{\sqrt{x^2 + 3x} + x} \\ &= \frac{3}{\sqrt{1 + 3/x} + 1}.\end{aligned}$$

Therefore,

$$\lim_{x \rightarrow +\infty} (\sqrt{x^2 + 3x} - x) = \frac{3}{2}.$$

$$3/2$$

Problem 5(c): combine the fractions first

$$\frac{1}{x} - \frac{1}{e^x - 1} = \frac{e^x - 1 - x}{x(e^x - 1)}.$$

The quotient is 0/0. After one application,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{e^x - 1 + xe^x},$$

which is still 0/0.

Differentiating again gives

$$\lim_{x \rightarrow 0} \frac{e^x}{2e^x + xe^x} = \frac{1}{2}.$$

$$1/2$$

Problem 6: indeterminate powers

Problem 6

Calculate:

$$(a) \quad \lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x},$$

$$(b) \quad \lim_{x \rightarrow 0} \left(\cos x + \frac{x^2}{2} \right)^{1/x^2},$$

$$(c) \quad \lim_{x \rightarrow 0^+} x^x.$$

Problem 6(a): take logarithms

Let

$$y = (\sin x + \cos x)^{1/x}.$$

Then

$$\ln y = \frac{\ln(\sin x + \cos x)}{x}.$$

L'Hospital gives

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\cos x - \sin x}{\sin x + \cos x} = 1.$$

$$y \rightarrow e$$

Problem 6(b): the logarithmic limit is zero

Put

$$L = \lim_{x \rightarrow 0} \frac{\ln(\cos x + x^2/2)}{x^2}.$$

After one application,

$$L = \lim_{x \rightarrow 0} \frac{x - \sin x}{2x(\cos x + x^2/2)}.$$

Differentiating again,

$$L = \lim_{x \rightarrow 0} \frac{1 - \cos x}{2(\cos x + x^2/2) + 2x(x - \sin x)} = 0.$$

The original limit is $e^0 = 1$.

Problem 6(c): the exponent wins slowly

Let $y = x^x$. Then

$$\ln y = x \ln x.$$

From Problem 5(a),

$$x \ln x \longrightarrow 0.$$

Therefore,

$$y \longrightarrow e^0 = 1.$$

$$\lim_{x \rightarrow 0^+} x^x = 1$$

Problem 7: count all real solutions

Problem 7

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Define

$$f(x) = 3x^5 - 5x^3 - 1.$$

We will locate its monotonicity intervals and then use continuity.

Problem 7: build the derivative sign chart

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x - 1)(x + 1).$$

The critical points are -1 , 0 and 1 .

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
$f'(x)$	+	−	−	+
f	increasing	decreasing	decreasing	increasing

The zero of f' at 0 does not change the sign.

Problem 7: evaluate the turning levels

$$\lim_{x \rightarrow -\infty} f(x) = -\infty,$$

$$f(-1) = 1,$$

$$f(0) = -1,$$

$$f(1) = -3,$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty.$$

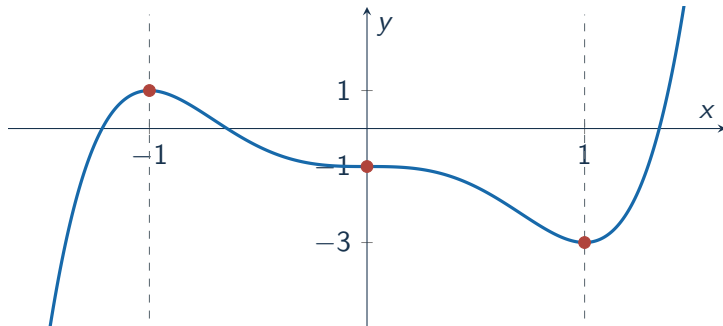
Thus there is exactly one zero in each interval

$$(-\infty, -1), \quad (-1, 0), \quad (1, +\infty).$$

There is no additional zero in $(0, 1)$ because $f < 0$ there.

The equation has exactly three real solutions.

The graph confirms the three crossings



A final method checklist

- For Rolle or Mean Value problems: verify hypotheses, compute the secant slope, solve for c , and check $c \in (a, b)$.
- For uniqueness: combine continuity with strict monotonicity.
- For L'Hospital: identify the form before differentiating.
- For $0 \cdot \infty$, $\infty - \infty$ and powers: rewrite first.
- When counting roots: use derivative signs and endpoint values together.