

Calculus 1

Week 4

Problem 1. Suppose that f is continuous on $[-7, 0]$, differentiable on $(-7, 0)$, $f(-7) = -3$, and $f'(x) \leq 2$ for every $x \in (-7, 0)$. Find the largest possible value of $f(0)$.

Solution. The Mean Value Theorem gives a point $c \in (-7, 0)$ such that

$$f(0) - f(-7) = f'(c)(0 - (-7)).$$

Using $f(-7) = -3$ and $f'(c) \leq 2$, we obtain

$$f(0) + 3 = 7f'(c) \leq 14.$$

Therefore,

$$f(0) \leq 11.$$

This bound is attained by the function $f(x) = 2x + 11$, which satisfies $f(-7) = -3$ and $f'(x) = 2$. Hence, the largest possible value is 11.

Problem 2. Show that

$$f(x) = x^3 - 7x^2 + 25x + 8$$

has exactly one real root.

Solution. First,

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

Since f is continuous, the Intermediate Value Theorem guarantees at least one real root.

For uniqueness, differentiate:

$$f'(x) = 3x^2 - 14x + 25.$$

The discriminant of this quadratic is

$$\Delta = (-14)^2 - 4 \cdot 3 \cdot 25 = 196 - 300 = -104 < 0.$$

Its leading coefficient is positive, so $f'(x) > 0$ for every $x \in \mathbb{R}$. Consequently, f is strictly increasing and can have at most one real root. Combining existence and uniqueness, f has exactly one real root.

Problem 3. Discuss the differentiability on $\mathbb{R}$ of

$$f(x) = \ln(1 + |x - 1|).$$

Solution. Using the definition of the absolute value,

$$f(x) = \begin{cases} \ln(2 - x), & x < 1, \\ \ln x, & x \geq 1. \end{cases}$$

Both expressions are well defined on the indicated intervals, and the function is continuous at 1 because

$$\lim_{x \rightarrow 1^-} \ln(2-x) = 0 = \lim_{x \rightarrow 1^+} \ln x = f(1).$$

Away from 1,

$$f'(x) = \begin{cases} -\frac{1}{2-x}, & x < 1, \\ \frac{1}{x}, & x > 1. \end{cases}$$

The one-sided derivatives at 1 are

$$f'_-(1) = -1, \quad f'_+(1) = 1.$$

They are different, so $f'(1)$ does not exist. Thus, f is differentiable on $\mathbb{R} \setminus \{1\}$ and has a corner at $x = 1$.

Problem 4. Consider

$$f(x) = \begin{cases} x^2, & x \leq 1, \\ 2, & x > 1, \end{cases} \quad g(x) = \begin{cases} x^2, & x \leq 1, \\ 1, & x > 1. \end{cases}$$

- (a) Sketch the graphs of f and g .
- (b) Determine whether each function is continuous at $x = 1$.
- (c) Determine whether each function is differentiable at $x = 1$.

Solution. For f ,

$$\lim_{x \rightarrow 1^-} f(x) = 1, \quad \lim_{x \rightarrow 1^+} f(x) = 2.$$

The one-sided limits are different, so f is not continuous at 1. Therefore, f is not differentiable there.

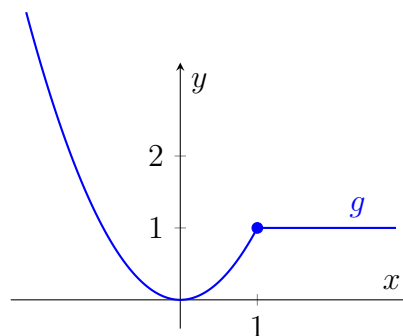
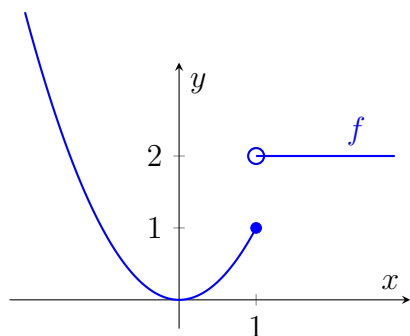
For g ,

$$\lim_{x \rightarrow 1^-} g(x) = 1 = \lim_{x \rightarrow 1^+} g(x) = g(1),$$

so g is continuous at 1. However,

$$g'_-(1) = \lim_{x \rightarrow 1^-} 2x = 2, \quad g'_+(1) = 0.$$

Thus, g is not differentiable at 1.



Problem 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x|x|$. Calculate $f'(x)$ for $x \neq 0$ and decide whether $f'(0)$ exists.

Solution. We write

$$f(x) = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

Hence,

$$f'(x) = \begin{cases} -2x, & x < 0, \\ 2x, & x > 0. \end{cases}$$

At the origin, use the definition:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h|h|}{h} = \lim_{h \rightarrow 0} |h| = 0.$$

Therefore, f is differentiable at 0 and

$$f'(x) = 2|x|, \quad x \in \mathbb{R}.$$

Problem 6. The position of a particle is

$$s(t) = \frac{1}{3}t^3 - t^2 - 3t.$$

- (a) Find the velocity and acceleration.
- (b) When is the velocity zero?
- (c) When is the acceleration zero?

Solution. The velocity and acceleration are

$$v(t) = s'(t) = t^2 - 2t - 3 = (t - 3)(t + 1),$$

and

$$a(t) = v'(t) = 2t - 2.$$

Thus, $v(t) = 0$ at $t = -1$ and $t = 3$. If time is restricted to $t \geq 0$, the physical stopping time is $t = 3$. Finally, $a(t) = 0$ at $t = 1$.

Problem 7. Consider

$$f(x) = \begin{cases} x + 2, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad g(x) = \begin{cases} x + 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 2, \quad \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = 1.$$

Does this contradict L'Hospital's rule?

Solution. For $x \neq 0$,

$$\frac{f(x)}{g(x)} = \frac{x+2}{x+1},$$

so

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 2.$$

Also, on a deleted neighbourhood of 0,

$$f'(x) = 1, \quad g'(x) = 1,$$

and therefore the derivative quotient tends to 1.

There is no contradiction. As $x \rightarrow 0$, we have $f(x) \rightarrow 2$ and $g(x) \rightarrow 1$. The original quotient is neither of the form $0/0$ nor of the form ∞/∞ . The values $f(0) = g(0) = 0$ do not change these limits. Consequently, the hypotheses of L'Hospital's rule are not satisfied.

Problem 8. Calculate the following limits:

(a) $\lim_{x \rightarrow +\infty} \frac{e^x}{x};$

(e) $\lim_{x \rightarrow 0^+} x \ln x;$

(b) $\lim_{x \rightarrow +\infty} (e^x - x);$

(f) $\lim_{x \rightarrow 0^+} x^x;$

(c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x};$

(d) $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right);$

(g) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}.$

Solution. (a) The quotient has the form ∞/∞ . Thus,

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x} = \lim_{x \rightarrow +\infty} e^x = +\infty.$$

(b) Factor e^x :

$$e^x - x = e^x \left(1 - \frac{x}{e^x} \right).$$

Since $x/e^x \rightarrow 0$, the difference tends to $+\infty$.

(c) L'Hospital's rule gives

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} e^x = 1.$$

(d) Combine the fractions:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}.$$

The quotient has the form $0/0$. Applying L'Hospital's rule twice,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{-\sin x}{2 \cos x - x \sin x} = 0. \end{aligned}$$

(e) Rewrite the product as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

Then

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

(f) Put $y = x^x$. Since

$$\ln y = x \ln x \longrightarrow 0,$$

we obtain $y \rightarrow e^0 = 1$.

(g) The quotient is $0/0$, so

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} (-\sin x) = 0.$$

Problem 9. Use the Taylor polynomial of degree 2 for $f(x) = \sqrt{x}$, centred at 16, to approximate $\sqrt{16.06}$. Give an upper bound for the error.

Solution. We have

$$f(16) = 4, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4x^{3/2}}.$$

Therefore,

$$f'(16) = \frac{1}{8}, \quad f''(16) = -\frac{1}{256},$$

and

$$T_2(x) = 4 + \frac{x - 16}{8} - \frac{(x - 16)^2}{512}.$$

At $x = 16.06$,

$$\sqrt{16.06} \approx T_2(16.06) = 4 + \frac{0.06}{8} - \frac{0.06^2}{512} = 4.00749296875.$$

To estimate the error, note that

$$f'''(x) = \frac{3}{8x^{5/2}}.$$

This function is decreasing on $[16, 16.06]$, so

$$|f'''(x)| \leq \frac{3}{8 \cdot 16^{5/2}} = \frac{3}{8192}.$$

Taylor's estimate gives

$$|R_2(16.06)| \leq \frac{3}{8192} \frac{0.06^3}{3!} < 1.32 \times 10^{-8}.$$

Problem 10.

(a) Find the tangent and normal lines to $f(x) = \ln(x^2 + 3x + 1) + e^{x^2+1}$ at $x = 0$.

(b) Find the tangent line to $g(x) = 7^x + 4e^x$ at $x = 0$.

Solution. (a) We have

$$f(0) = \ln 1 + e = e.$$

Moreover,

$$f'(x) = \frac{2x + 3}{x^2 + 3x + 1} + 2xe^{x^2+1},$$

so $f'(0) = 3$. The tangent line is

$$y - e = 3x, \quad \text{or} \quad y = 3x + e.$$

The normal slope is $-1/3$, and hence the normal line is

$$y - e = -\frac{1}{3}x.$$

(b) We have

$$g(0) = 1 + 4 = 5$$

and

$$g'(x) = 7^x \ln 7 + 4e^x.$$

Thus, $g'(0) = \ln 7 + 4$, and the tangent line is $y = 5 + (\ln 7 + 4)x$.

Problem 11. Find the derivative of each function and compare the domain of the derivative with the domain of the original function.

(a) $f(x) = \ln \left(\frac{x^2 + 3x}{x^3 + 1} \right);$

(b) $f(x) = \frac{e^x \cos x}{\ln^2 x};$

(c) $f(x) = \frac{\sqrt{\ln(x+2)}}{x^2};$

(d) $f(x) = \frac{\sqrt{e^x + 1}}{x^2 + \sin^2 x};$

(e) $f(x) = \ln \sqrt{e^x x^3};$

(f) $f(x) = \sqrt{x \ln^2 x};$

(g) $f(x) = x^{2(x+1)};$

(h) $f(x) = (x + 1)^{\cos^2 x}.$

Solution. (a) A sign chart for

$$\frac{x(x+3)}{(x+1)(x^2-x+1)}$$

gives

$$\text{Dom}(f) = (-3, -1) \cup (0, +\infty).$$

On this domain,

$$f'(x) = \frac{2x+3}{x^2+3x} - \frac{3x^2}{x^3+1}.$$

The derivative has the same domain as f .

(b) The domain is

$$(0, 1) \cup (1, +\infty).$$

Using the quotient rule,

$$f'(x) = \frac{e^x(\cos x - \sin x)(\ln x)^2 - e^x \cos x \frac{2 \ln x}{x}}{(\ln x)^4}.$$

The derivative has the same domain.

(c) The square root requires $\ln(x+2) \geq 0$, so $x \geq -1$, and the denominator requires $x \neq 0$. Hence,

$$\text{Dom}(f) = [-1, 0) \cup (0, +\infty).$$

For $x > -1$, $x \neq 0$,

$$f'(x) = \frac{1}{2x^2(x+2)\sqrt{\ln(x+2)}} - \frac{2\sqrt{\ln(x+2)}}{x^3}.$$

The derivative is not finite at $x = -1$, so

$$\text{Dom}(f') = (-1, 0) \cup (0, +\infty).$$

(d) The numerator is defined for every real x , while $x^2 + \sin^2 x = 0$ only at $x = 0$. Thus,

$$\text{Dom}(f) = \mathbb{R} \setminus \{0\}.$$

The derivative is

$$f'(x) = \frac{(x^2 + \sin^2 x) \frac{e^x}{2\sqrt{e^x+1}} - \sqrt{e^x+1} (2x + 2 \sin x \cos x)}{(x^2 + \sin^2 x)^2}.$$

Its domain is also $\mathbb{R} \setminus \{0\}$.

(e) The logarithm requires $e^x x^3 > 0$, which is equivalent to $x > 0$. On this domain,

$$f(x) = \frac{1}{2} \ln(e^x x^3) = \frac{x}{2} + \frac{3}{2} \ln x,$$

so

$$f'(x) = \frac{1}{2} + \frac{3}{2x}, \quad x > 0.$$

(f) The domain is $(0, +\infty)$. For $x \neq 1$,

$$f'(x) = \frac{\ln^2 x + 2 \ln x}{2\sqrt{x \ln^2 x}}.$$

Since $\sqrt{x \ln^2 x} = \sqrt{x} |\ln x|$, the function has a corner at $x = 1$. Therefore,

$$\text{Dom}(f') = (0, 1) \cup (1, +\infty).$$

(g) We use logarithmic differentiation on $x > 0$:

$$\ln f(x) = 2(x+1) \ln x.$$

Thus,

$$f'(x) = x^{2(x+1)} \left(2 \ln x + \frac{2(x+1)}{x} \right), \quad x > 0.$$

The function and its derivative have the same natural real domain $(0, +\infty)$.

(h) On the natural interval $x > -1$,

$$\ln f(x) = \cos^2 x \ln(x+1).$$

Therefore,

$$f'(x) = (x+1)^{\cos^2 x} \left(-2 \sin x \cos x \ln(x+1) + \frac{\cos^2 x}{x+1} \right).$$

Both domains are $(-1, +\infty)$.

Problem 12. Determine the values of $a, b \in \mathbb{R}$ for which

$$f(x) = \begin{cases} \sin x, & x < \frac{\pi}{2}, \\ ax + b, & x \geq \frac{\pi}{2} \end{cases}$$

is continuous and differentiable on $\mathbb{R}$.

Solution. Continuity at $x = \pi/2$ requires

$$\lim_{x \rightarrow (\pi/2)^-} \sin x = a \frac{\pi}{2} + b.$$

Thus,

$$a \frac{\pi}{2} + b = 1.$$

For differentiability, the one-sided derivatives must agree:

$$\lim_{x \rightarrow (\pi/2)^-} \cos x = a.$$

Hence $a = 0$. Substitution into the continuity condition gives $b = 1$. Therefore, $a = 0$ and $b = 1$.

Problem 13. Use L'Hospital's rule to calculate:

- (a) $\lim_{x \rightarrow 0^+} \frac{e^x - 1}{\cos x - 1}$;
- (b) $\lim_{x \rightarrow +\infty} \frac{\sqrt{x+1}}{\ln^2(x+1)}$;
- (c) $\lim_{x \rightarrow +\infty} \frac{\ln x}{e^x}$;
- (d) $\lim_{x \rightarrow +\infty} \frac{x^2 + 3x}{\ln(x^3 + 1)}$;
- (e) $\lim_{t \rightarrow +\infty} t \ln \left(1 + \frac{3}{t} \right)$.

Solution. (a) The form is $0/0$. Thus,

$$\lim_{x \rightarrow 0^+} \frac{e^x - 1}{\cos x - 1} = \lim_{x \rightarrow 0^+} \frac{e^x}{-\sin x} = -\infty.$$

(b) The form is ∞/∞ . First,

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{x+1}}{\ln^2(x+1)} = \lim_{x \rightarrow +\infty} \frac{1/(2\sqrt{x+1})}{2\ln(x+1)/(x+1)} = \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{\sqrt{x+1}}{\ln(x+1)}.$$

The new quotient is still ∞/∞ . Applying the rule again,

$$\frac{1}{4} \lim_{x \rightarrow +\infty} \frac{1/(2\sqrt{x+1})}{1/(x+1)} = \frac{1}{8} \lim_{x \rightarrow +\infty} \sqrt{x+1} = +\infty.$$

(c) We obtain

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{e^x} = \lim_{x \rightarrow +\infty} \frac{1/x}{e^x} = \lim_{x \rightarrow +\infty} \frac{1}{xe^x} = 0.$$

(d) Applying L'Hospital's rule,

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 3x}{\ln(x^3 + 1)} = \lim_{x \rightarrow +\infty} \frac{2x + 3}{3x^2/(x^3 + 1)} = \lim_{x \rightarrow +\infty} \frac{(2x + 3)(x^3 + 1)}{3x^2} = +\infty.$$

(e) Rewrite the product as a quotient:

$$t \ln \left(1 + \frac{3}{t} \right) = \frac{\ln(1 + 3/t)}{1/t}.$$

This is $0/0$. Therefore,

$$\lim_{t \rightarrow +\infty} \frac{\ln(1 + 3/t)}{1/t} = \lim_{t \rightarrow +\infty} \frac{-3/[t^2(1 + 3/t)]}{-1/t^2} = \lim_{t \rightarrow +\infty} \frac{3}{1 + 3/t} = 3.$$

Problem 14. Let f be the continuous piecewise linear function whose graph joins the points

$$(-4, 5), \quad (-1, 2), \quad (2, 4), \quad (4, 0)$$

in this order. Sketch the graph of f' .

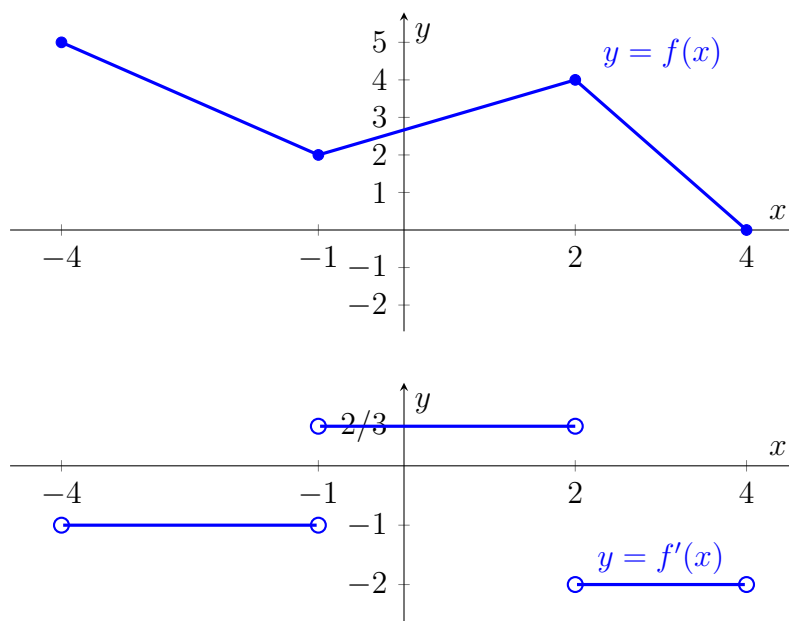
Solution. On each open segment, the derivative equals the slope of that segment. The slopes are

$$\begin{aligned}\frac{2-5}{-1-(-4)} &= -1, \\ \frac{4-2}{2-(-1)} &= \frac{2}{3}, \\ \frac{0-4}{4-2} &= -2.\end{aligned}$$

Therefore,

$$f'(x) = \begin{cases} -1, & -4 < x < -1, \\ \frac{2}{3}, & -1 < x < 2, \\ -2, & 2 < x < 4. \end{cases}$$

The derivative does not exist at the corners $x = -1$ and $x = 2$. With the usual two-sided definition, it is also not defined at the endpoints -4 and 4 .



Problem 15. The position of an object at time t is

$$s(t) = \frac{t+1}{t+4}.$$

Find its velocity. Does the object ever stop moving?

Solution. By the quotient rule,

$$v(t) = s'(t) = \frac{(t+4) - (t+1)}{(t+4)^2} = \frac{3}{(t+4)^2}, \quad t \neq -4.$$

This expression is strictly positive everywhere on its domain. Therefore, the object never stops moving.