

Calculus 1

Lecture 9: Extrema and the First Derivative Test

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What we will be able to do today

- Distinguish local extrema from absolute extrema.
- Use the Extreme Value Theorem and Fermat's theorem correctly.
- Find every critical number of a function.
- Read increasing and decreasing behaviour from the sign of f' .
- Find local and absolute extrema with a reliable procedure.

A review: Local and absolute extrema

Absolute extrema compare the whole domain

Let $f : D \rightarrow \mathbb{R}$ and let $a \in D$.

f has an **absolute maximum** at a if

$$f(x) \leq f(a) \quad \text{for every } x \in D.$$

f has an **absolute minimum** at a if

$$f(a) \leq f(x) \quad \text{for every } x \in D.$$

The value $f(a)$ is the extremum, and a is a point where it is attained.

Local extrema compare nearby points

f has a **local maximum** at a if, for some $\delta > 0$,

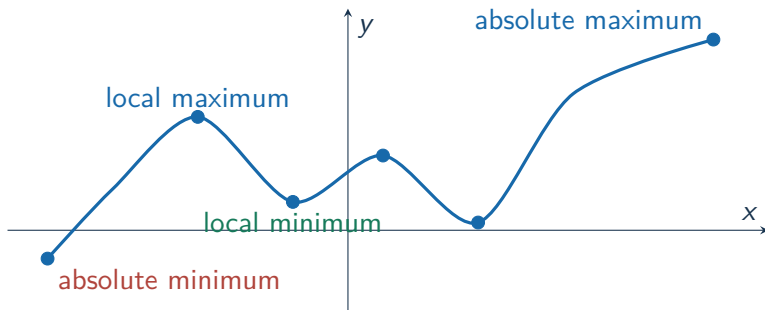
$$f(x) \leq f(a) \quad \text{whenever } x \in D \text{ and } |x - a| < \delta.$$

f has a **local minimum** at a if, for some $\delta > 0$,

$$f(a) \leq f(x) \quad \text{whenever } x \in D \text{ and } |x - a| < \delta.$$

Only a sufficiently small neighbourhood of a enters the comparison.

One graph can contain several kinds of extrema



An absolute extremum may occur at an endpoint.

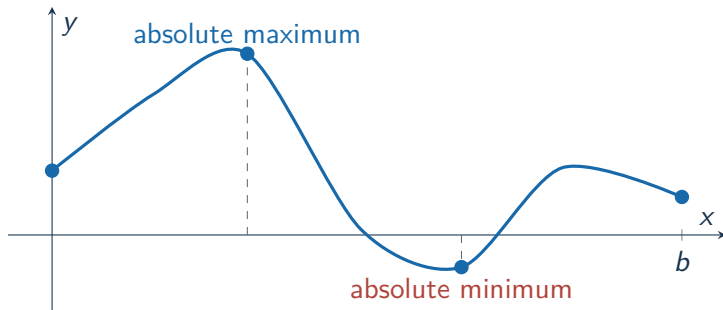
The Extreme Value Theorem guarantees attainment

If f is continuous on a closed bounded interval $[a, b]$, then f attains

- an absolute maximum on $[a, b]$;
- an absolute minimum on $[a, b]$.

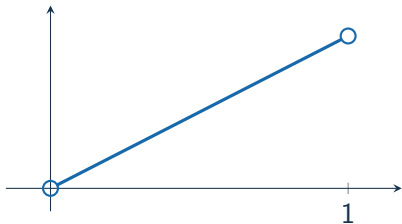
The theorem answers an existence question. It does not locate the extrema.

A continuous graph on $[a, b]$ reaches both extremes

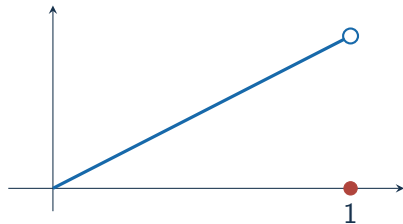


Closed endpoints and continuity prevent the graph from losing its extreme values.

The hypotheses of the theorem matter



$f(x) = x$ on $(0, 1)$ has neither a maximum nor a minimum.



$f(x) = x$ for $x < 1$ and $f(1) = 0$ has no maximum on $[0, 1]$.

Critical numbers and derivative signs

Fermat's theorem supplies candidates

Suppose that c is an interior point of the domain.

If f is differentiable at c and has a local extremum there, then

$$f'(c) = 0.$$

The conclusion is necessary, but it is not sufficient:

$$f'(c) = 0 \quad \nRightarrow \quad f \text{ has an extremum at } c.$$

Why a differentiable extremum has slope zero

Suppose that f has a local maximum at the interior point c .

For $x < c$ near c ,

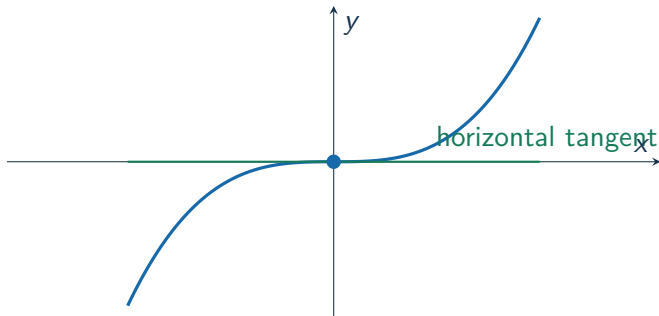
$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

For $x > c$ near c ,

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

If the derivative exists, both one-sided limits agree. Their common value is 0.

A horizontal tangent may fail to be an extremum



For $f(x) = x^3$, we have $f'(0) = 0$, but f increases on both sides of 0.

Critical numbers include corners and cusps

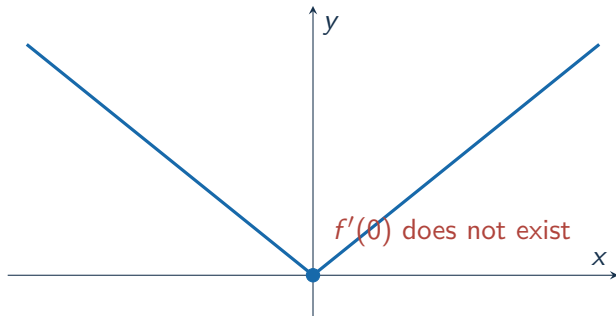
A number c in the domain of f is a **critical number** if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Two details are essential:

- c must belong to the domain of f ;
- a critical number is only a candidate for an extremum.

A corner can be a critical minimum



The number 0 is critical for $f(x) = |x|$, and f has a local and absolute minimum there.

The sign of f' determines monotonicity

Let f be differentiable on an interval I .

$$f'(x) > 0 \text{ on } I \implies f \text{ is strictly increasing on } I,$$

$$f'(x) < 0 \text{ on } I \implies f \text{ is strictly decreasing on } I,$$

$$f'(x) = 0 \text{ on } I \implies f \text{ is constant on } I.$$

This follows directly from the Mean Value Theorem.

The first derivative test reads a sign change

Let c be a critical number and suppose that f is continuous at c .

sign of f' before c	sign of f' after c	conclusion
+	−	local maximum
−	+	local minimum
+	+	no local extremum
−	−	no local extremum

A reliable first derivative analysis

1. Determine the domain of f .
2. Calculate f' .
3. Find the critical numbers of f .
4. Use them to divide the domain into intervals.
5. Determine the sign of f' on every interval.
6. Read monotonicity and local extrema from the sign chart.

Example 1: begin with a quadratic

Find the local and absolute extrema of

$$f(x) = x^2 - 4x + 1, \quad x \in \mathbb{R}.$$

Differentiate and find the critical numbers:

$$f'(x) = 2x - 4 = 2(x - 2), \quad f'(x) = 0 \iff x = 2.$$

x	$(-\infty, 2)$	$(2, +\infty)$
$f'(x)$	$-$	$+$
f	$\searrow$	$\nearrow$

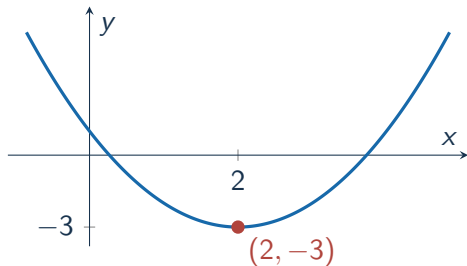
Example 1: the minimum is absolute

The sign changes from $-$ to $+$ at $x = 2$. Moreover,

$$f(2) = -3,$$

$$f(x) = (x - 2)^2 - 3 \geq -3.$$

Absolute minimum: -3 at
 $x = 2$.
No maximum.



Example 2: identify the critical numbers

Consider

$$f(x) = x^3 - 3x + 1.$$

Differentiate and factor:

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x - 1)(x + 1). \end{aligned}$$

Therefore,

$$f'(x) = 0 \quad \Longleftrightarrow \quad x = -1 \text{ or } x = 1.$$

Example 2: determine the sign of the derivative

The factors $x + 1$ and $x - 1$ change sign at -1 and 1 .

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$x + 1$	$-$	$+$	$+$
$x - 1$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$

Hence, f increases, then decreases, then increases.

Example 2: classify the critical numbers

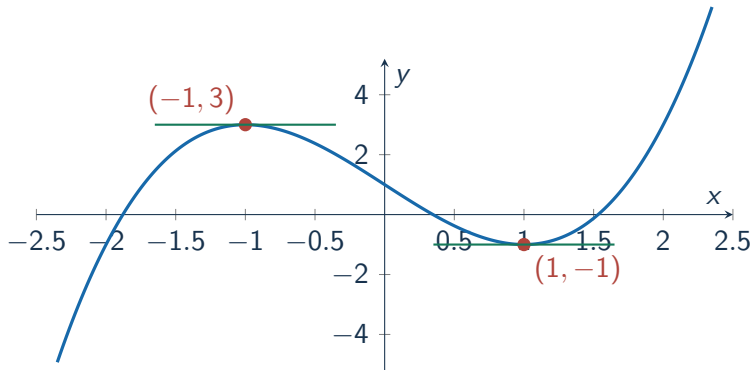
At $x = -1$, the sign of f' changes from $+$ to $-$. Therefore, f has a local maximum there:

$$f(-1) = (-1)^3 - 3(-1) + 1 = 3.$$

At $x = 1$, the sign of f' changes from $-$ to $+$. Therefore, f has a local minimum there:

$$f(1) = 1^3 - 3(1) + 1 = -1.$$

Example 2: the derivative signs predict the graph



The extrema are local only because the cubic is unbounded above and below.

Example 3: the domain comes first

Find the local and absolute extrema of

$$f(x) = x + \frac{4}{x}.$$

The natural domain is

$$\text{Dom}(f) = \mathbb{R} \setminus \{0\}.$$

Differentiate and factor:

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x - 2)(x + 2)}{x^2}.$$

The critical numbers are -2 and 2 . The number 0 is not critical because it does not belong to the domain.

Example 3: the excluded point also separates intervals

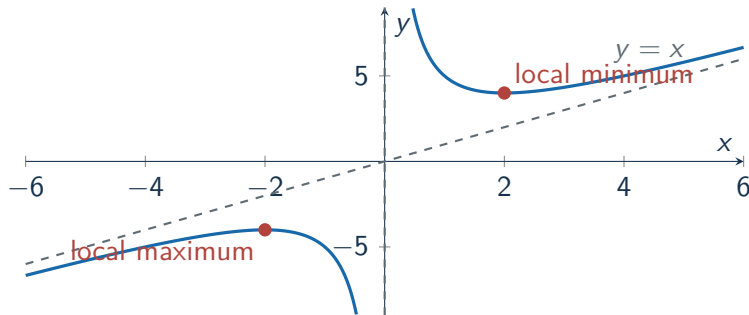
Since $x^2 > 0$ on the domain, the numerator determines the sign of f' .

x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	+	-	-	+
f				

Thus,

$f(-2) = -4$ is a local maximum, $f(2) = 4$ is a local minimum.

Example 3: local extrema need not be absolute



The function is unbounded above and below. Therefore, it has no absolute extrema.

Absolute extrema on a closed interval

Absolute extrema have three types of candidates

Let f be continuous on $[a, b]$.

An absolute extremum may occur at

- an interior critical number;
- the left endpoint a ;
- the right endpoint b .

Fermat's theorem explains the interior candidates. The Extreme Value Theorem guarantees that the comparison will find both extrema.

The closed-interval method

To find the absolute extrema of a continuous function f on $[a, b]$:

1. Calculate f' and find every critical number in (a, b) .
2. Evaluate f at those critical numbers and at a and b .
3. Compare all values.

The largest value is the absolute maximum. The smallest is the absolute minimum.

Example 4: split the absolute value first

Find the absolute extrema of

$$f(x) = \left| x^2 - 1 \right| + \frac{x}{2} \quad \text{on } [-2, 2].$$

The expression inside the absolute value changes sign at $x = -1$ and $x = 1$. Therefore,

$$f(x) = \begin{cases} x^2 - 1 + \frac{x}{2}, & -2 \leq x \leq -1, \\ 1 - x^2 + \frac{x}{2}, & -1 < x < 1, \\ x^2 - 1 + \frac{x}{2}, & 1 \leq x \leq 2. \end{cases}$$

The points -1 and 1 must be checked separately.

Example 4: find every critical number

On the three open pieces,

$$f'(x) = \begin{cases} 2x + \frac{1}{2}, & -2 < x < -1, \\ -2x + \frac{1}{2}, & -1 < x < 1, \\ 2x + \frac{1}{2}, & 1 < x < 2. \end{cases}$$

The derivative does not exist at $x = -1$ and $x = 1$. Inside $(-1, 1)$, $-2x + \frac{1}{2} = 0$ exactly when $x = \frac{1}{4}$.

Although $2x + \frac{1}{2} = 0$ gives $x = -1/4$, that value does not lie in either interval where this formula applies. Thus, the critical numbers are

$$\boxed{-1, \frac{1}{4}, 1.}$$

Example 4: classify the local extrema

x	$(-2, -1)$	$(-1, 1/4)$	$(1/4, 1)$	$(1, 2)$
$f'(x)$	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

The sign changes show that

- $x = -1$ and $x = 1$ are local minimum points;
- $x = 1/4$ is a local maximum point.

These conclusions are local. To find the absolute extrema, we must also test the endpoints.

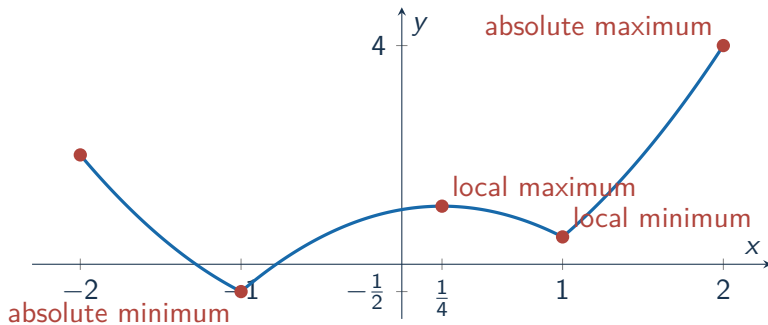
Example 4: compare every candidate value

x	$f(x)$	role
-2	2	endpoint
-1	$-1/2$	critical number
$1/4$	$17/16$	critical number
1	$1/2$	critical number
2	4	endpoint

Absolute minimum: $-1/2$ at $x = -1$.

Absolute maximum: 4 at $x = 2$.

Example 4: the graph confirms every comparison



A compact decision process

Local and absolute questions use different comparisons

Local extrema

- Find critical numbers.
- Make a sign chart for f' .
- Look for a change of sign.

Absolute extrema on $[a, b]$

- Find interior critical numbers.
- Include both endpoints.
- Compare the function values.

What to remember

- The Extreme Value Theorem guarantees absolute extrema on $[a, b]$.
- Fermat's theorem produces candidates, not conclusions.
- Critical numbers belong to the domain of the function.
- The sign of f' describes increasing and decreasing intervals.
- On a closed interval, compare every interior critical value with both endpoint values.