

Calculus 1

Lecture 10: Concavity, Asymptotes and Graph Analysis

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What we will be able to do today

- Read concavity from the sign of the second derivative.
- Distinguish critical points from inflection points.
- Use the second derivative to classify local extrema.
- Find vertical, horizontal and oblique asymptotes.
- Combine limits, f' and f'' in a complete graph analysis.

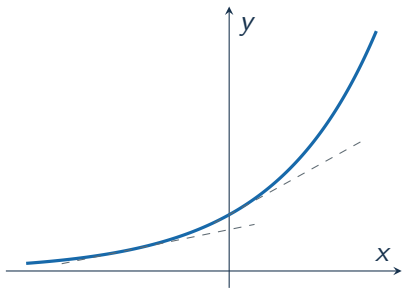
The second derivative and the shape of
a graph

The two derivatives answer different questions

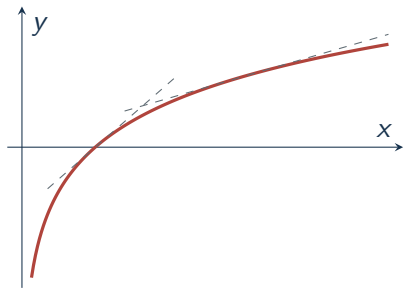
	Sign	Meaning for the graph
f'	$f' > 0$ or $f' < 0$	rising or falling
f''	$f'' > 0$ or $f'' < 0$	tangent slopes increase or decrease

Two graphs may both be increasing while bending in opposite ways.

Increasing functions can have different shapes



e^x : increasing slopes, so $f'' > 0$.



$\ln x$: decreasing slopes, so $f'' < 0$.

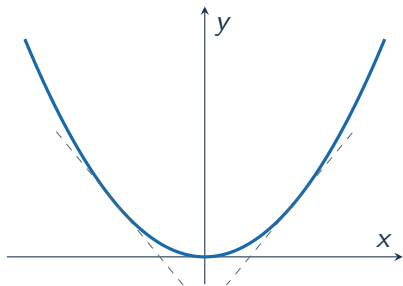
The second derivative criterion

Suppose that f is twice differentiable on an interval I .

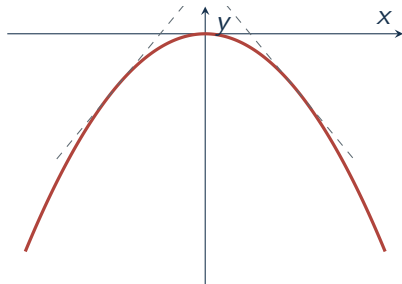
- If $f''(x) > 0$ on I , then f is **convex** on I .
- If $f''(x) < 0$ on I , then f is **concave** on I .

Indeed, $f'' > 0$ means that f' is increasing, so the tangent slopes become larger from left to right.

The sign of the second derivative changes the bending



Convex: $f'' > 0$ and slopes increase.



Concave: $f'' < 0$ and slopes decrease.

Inflection points and the second derivative test

An inflection point requires a change of concavity

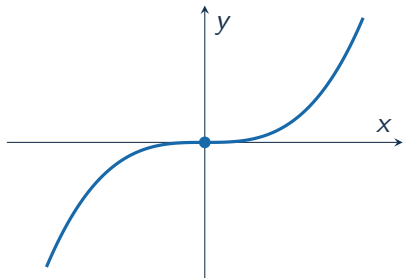
An interior point c is an **inflection point** when the graph changes from convex to concave, or from concave to convex, at c .

Solving

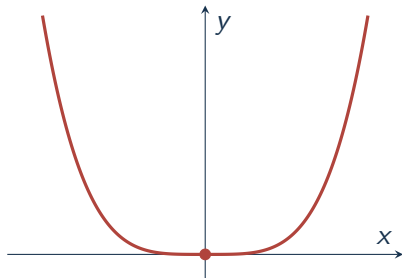
$$f''(x) = 0$$

produces only **candidates**. We must check the sign of f'' on both sides.

A zero of the second derivative may or may not work



x^3 : $f''(0) = 0$ and concavity changes.



x^4 : $f''(0) = 0$ but the graph stays convex.

A convex function lies above its tangent lines

If f is differentiable and convex, then

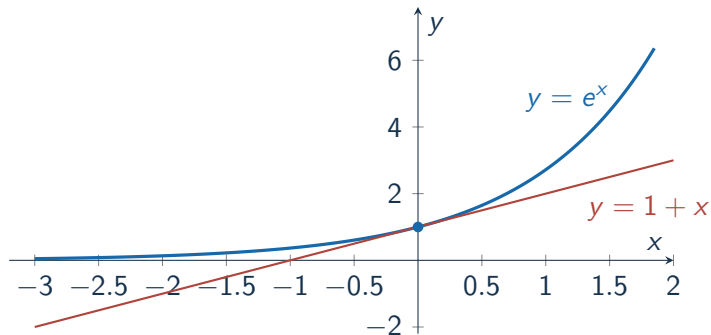
$$f(x) \geq f(a) + f'(a)(x - a).$$

For a concave function, the inequality reverses.

Since $f(x) = e^x$ satisfies $f''(x) = e^x > 0$, its tangent at $a = 0$ gives

$$e^x \geq 1 + x \quad \text{for every } x \in \mathbb{R}.$$

The inequality is visible on the graph



The graph of e^x never falls below its tangent at the origin.

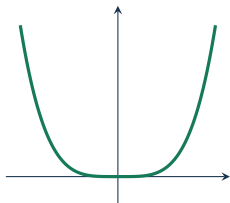
The second derivative test

Suppose that $f'(c) = 0$ and $f''(c)$ exists.

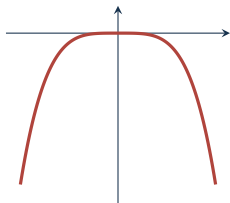
- If $f''(c) > 0$, then f has a local minimum at c .
- If $f''(c) < 0$, then f has a local maximum at c .
- If $f''(c) = 0$, the test gives no conclusion.

The case $f''(c) = 0$ is inconclusive

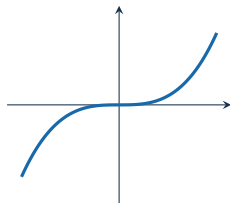
All three functions satisfy $f'(0) = f''(0) = 0$.



x^4 : minimum



$-x^4$: maximum



x^3 : neither

Example: the derivatives provide all candidates

Consider

$$f(x) = x^4 - 4x^2.$$

Then

$$f'(x) = 4x(x^2 - 2), \quad f''(x) = 12x^2 - 8.$$

The critical numbers satisfy $f'(x) = 0$:

$$x = -\sqrt{2}, \quad x = 0, \quad x = \sqrt{2}.$$

Example: classify the critical points

Evaluate the second derivative:

$$\begin{aligned}f''(0) &= -8 < 0, \\f''(\pm\sqrt{2}) &= 12(2) - 8 = 16 > 0.\end{aligned}$$

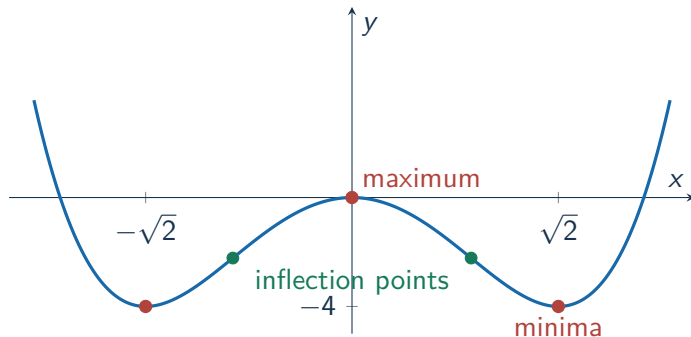
Therefore,

$(0, 0)$ is a local maximum

and

$(\pm\sqrt{2}, -4)$ are local minima.

Example: extrema and inflection points have different roles



Asymptotes

Vertical and horizontal asymptotes

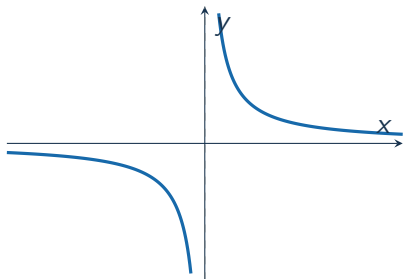
The line $x = a$ is a **vertical asymptote** if at least one one-sided limit at a equals $+\infty$ or $-\infty$.

The line $y = L$ is a **horizontal asymptote** as $x \rightarrow +\infty$ if

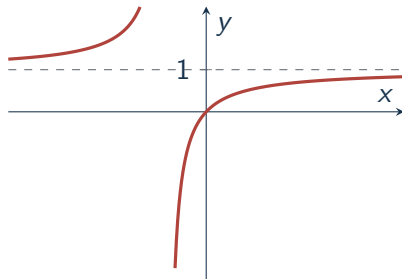
$$\lim_{x \rightarrow +\infty} f(x) = L.$$

The definition at $-\infty$ is analogous.

Vertical and horizontal asymptotes are limit statements



$x = 0$ is a vertical asymptote for $1/x$.



$y = 1$ is a horizontal asymptote for $x/(x+1)$.

Oblique asymptotes

At $+\infty$, the line $y = mx + n$, where $m \neq 0$, is an **oblique asymptote** if

$$\lim_{x \rightarrow +\infty} (f(x) - (mx + n)) = 0.$$

When the limits exist,

$$m = \lim_{x \rightarrow +\infty} \frac{f(x)}{x}, \quad n = \lim_{x \rightarrow +\infty} (f(x) - mx).$$

At $-\infty$, replace $+\infty$ by $-\infty$ in all three limits. The two ends may have different asymptotes.

Example: begin with the domain and vertical behaviour

Let

$$f(x) = \frac{x^2 + 1}{x - 1}.$$

The numerator is positive for every real x , while the denominator vanishes at $x = 1$.
Hence,

$$\lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty.$$

Vertical asymptote: $x = 1$

Example: polynomial division finds the oblique asymptote

Divide the polynomials:

$$\frac{x^2 + 1}{x - 1} = x + 1 + \frac{2}{x - 1}.$$

Since

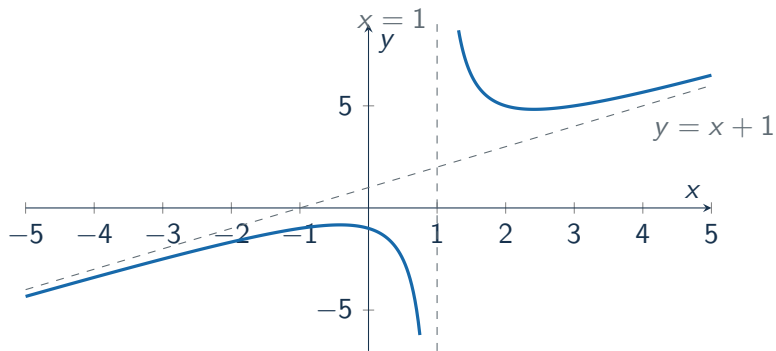
$$\frac{2}{x - 1} \rightarrow 0 \quad (x \rightarrow \pm\infty),$$

we obtain

$$\lim_{x \rightarrow \pm\infty} (f(x) - (x + 1)) = 0.$$

Oblique asymptote: $y = x + 1$

Example: the asymptotes organize the two branches



A systematic graphing procedure

The first three steps establish the large-scale shape

1. Find the domain and note symmetry or periodicity.
2. Find intercepts and determine signs when useful.
3. Calculate limits at boundary points and at infinity. Identify asymptotes.

These steps tell us where the graph exists and how its branches begin and end.

The derivatives complete the sketch

4. Use f' to find intervals of increase and decrease and local extrema.
5. Use f'' to find convexity, concavity and inflection points.
6. Combine all the information in a coherent sketch.

A graph should be the conclusion of the analysis, rather than a guess from a few points.

Keep the three kinds of special numbers separate

Boundary point

Study the corresponding limits. The function may be undefined at this point.

Critical number

$f'(c) = 0$ or $f'(c)$ does not exist, with c in the domain of f .

Inflection candidate

$f''(c) = 0$ or $f''(c)$ does not exist.

- An extremum requires a change in monotonicity.
- An inflection point requires a change in concavity and must belong to the graph.

Complete analysis of a rational function

The function to be analysed

Consider

$$f(x) = \frac{x^3}{x^2 - 1}.$$

We will determine

- the domain, symmetry and intercepts,
- all asymptotes and limiting behaviour,
- monotonicity and local extrema,
- concavity and inflection points.

Step 1: domain, symmetry and intercepts

The denominator vanishes at $x = \pm 1$, so

$$\text{Dom}(f) = \mathbb{R} \setminus \{-1, 1\}.$$

Moreover,

$$f(-x) = \frac{-x^3}{x^2 - 1} = -f(x).$$

Thus, f is odd and its graph is symmetric about the origin.

Finally, $f(x) = 0$ only when $x = 0$, so the only intercept is $(0, 0)$.

Step 2: behaviour near the excluded points

Since the numerator has a fixed sign near each excluded point,

$$\lim_{x \rightarrow -1^-} f(x) = -\infty,$$

$$\lim_{x \rightarrow -1^+} f(x) = +\infty,$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty,$$

$$\lim_{x \rightarrow 1^+} f(x) = +\infty.$$

Vertical asymptotes: $x = -1$ and $x = 1$

Step 3: behaviour at infinity

Polynomial division gives

$$f(x) = x + \frac{x}{x^2 - 1}.$$

Because

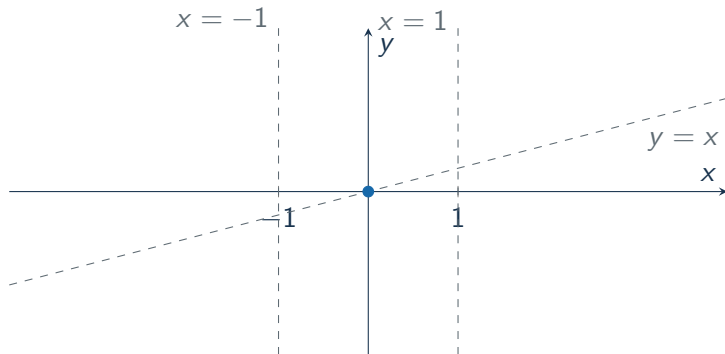
$$\frac{x}{x^2 - 1} \longrightarrow 0 \quad (x \rightarrow \pm\infty),$$

we have

$$f(x) - x \longrightarrow 0.$$

Oblique asymptote: $y = x$

The asymptotes determine the possible branches



Before differentiating, we already know the framework of the graph.

Step 4: calculate the first derivative

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{3x^2(x^2 - 1) - x^3(2x)}{(x^2 - 1)^2} \\ &= \frac{x^4 - 3x^2}{(x^2 - 1)^2} = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}. \end{aligned}$$

The critical numbers in the domain are

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

Step 5: the sign chart for the first derivative

The denominator is positive on the domain. Therefore, apart from the zero at $x = 0$, the sign comes from $x^2 - 3$.

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	+	-	-	-	-	+
f						

At $x = 0$, the derivative vanishes, but it is negative on both sides. The function continues to decrease through this stationary point.

Step 6: identify the local extrema

At $x = -\sqrt{3}$, the sign of f' changes from $+$ to $-$, so we obtain a local maximum:

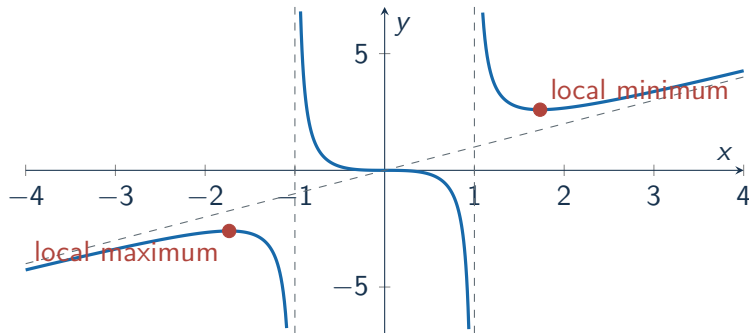
$$f(-\sqrt{3}) = -\frac{3\sqrt{3}}{2}.$$

At $x = \sqrt{3}$, the sign changes from $-$ to $+$, so we obtain a local minimum:

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{2}.$$

The stationary point $(0,0)$ is not an extremum.

The first derivative locates the turns of the outer branches



Step 7: calculate the second derivative

Differentiate the expression for f' and simplify:

$$f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Since

$$x^2 + 3 > 0$$

for every real x , the sign of f'' depends only on x and $x^2 - 1$.

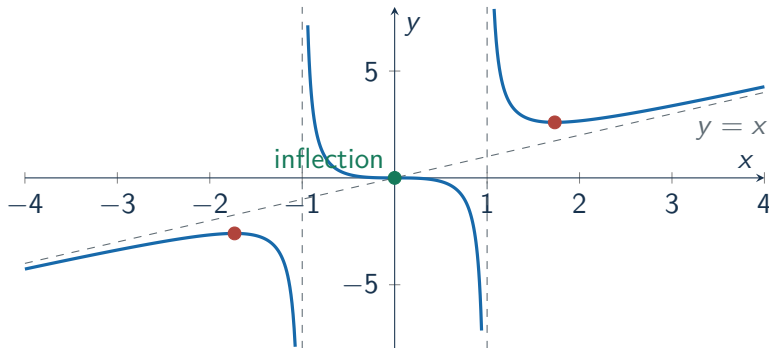
Step 8: the sign chart for the second derivative

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f''(x)$	$-$	$+$	$-$	$+$
shape	concave	convex	concave	convex

The concavity changes at $x = 0$, so $(0, 0)$ is an inflection point.

The concavity also changes across $x = \pm 1$, but those numbers are not in the domain and therefore do not produce inflection points.

The completed graph contains every part of the analysis



The complete analysis at a glance

Domain	$\mathbb{R} \setminus \{-1, 1\}$
Symmetry	Odd
Intercept	$(0, 0)$
Asymptotes	$x = -1$, $x = 1$ and $y = x$
Local maximum	$\left(-\sqrt{3}, -\frac{3\sqrt{3}}{2}\right)$
Local minimum	$\left(\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$
Inflection point	$(0, 0)$

What to remember

- The sign of f'' describes how tangent slopes change.
- A zero of f'' is only a candidate for an inflection point.
- The second derivative test classifies a critical point only when $f''(c) \neq 0$.
- Asymptotes come from limits and do not act as barriers.
- A reliable sketch combines domain, limits, f' and f'' .