

Calculus 1

Practice 5: Extrema, Asymptotes and Graph Sketching

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One method for all four problems

For each function, we will proceed in the same order:

1. determine the domain, symmetry and intercepts;
2. calculate the limits at the boundary of the domain and at infinity;
3. identify every vertical, horizontal or oblique asymptote;
4. use the sign of f' to locate local and absolute extrema;
5. draw the graph only after completing the analysis.

The graph is the conclusion of the calculation, not a guess.

Problem 1: a polynomial

Problem 1

Carry out a complete graph analysis of

$$f(x) = x^4 - 4x^2 + 3.$$

Find:

- the domain, symmetry, intercepts and limits at infinity;
- all local and absolute maxima and minima;
- all asymptotes, if any;
- the range.

Then sketch the graph.

Problem 1: basic information and end behaviour

The domain is $\mathbb{R}$, and

$$f(-x) = f(x),$$

so the graph is symmetric with respect to the y -axis.

Factoring gives

$$f(x) = (x^2 - 1)(x^2 - 3).$$

Hence, the x -intercepts are

$$x = \pm 1, \quad x = \pm\sqrt{3},$$

and the y -intercept is $(0, 3)$.

Since the leading term is x^4 ,

$$\lim_{x \rightarrow \pm\infty} f(x) = +\infty.$$

There are no asymptotes.

Problem 1: critical points

Differentiate and factor:

$$\begin{aligned}f'(x) &= 4x^3 - 8x \\&= 4x(x^2 - 2) \\&= 4x(x - \sqrt{2})(x + \sqrt{2}).\end{aligned}$$

Therefore, the critical points are

$$x = -\sqrt{2}, \quad x = 0, \quad x = \sqrt{2}.$$

They divide the real line into four intervals. We now determine the sign of f' on each interval.

Problem 1: sign chart and extrema

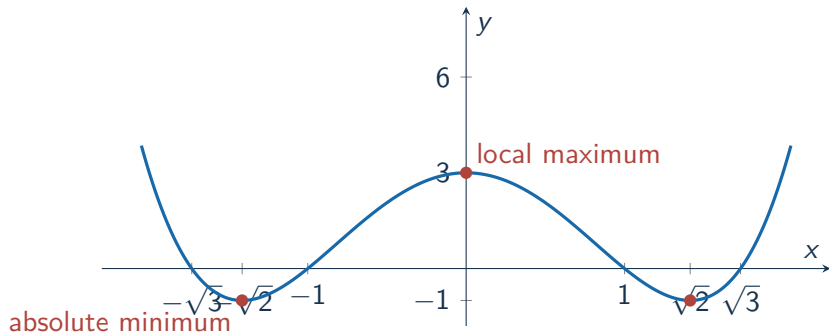
x	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, +\infty)$
$f'(x)$	$-$	$+$	$-$	$+$
f	decreasing	increasing	decreasing	increasing

The corresponding values are

$$f(\pm\sqrt{2}) = -1, \quad f(0) = 3.$$

$(0, 3)$ is a local maximum. The points $(\pm\sqrt{2}, -1)$ are absolute minima. There is no absolute maximum.

Problem 1: complete graph



$$f(\mathbb{R}) = [-1, +\infty).$$

Problem 2: a bounded rational function

Problem 2

Carry out a complete graph analysis of

$$f(x) = \frac{x}{x^2 + 1}.$$

Find:

- the domain, symmetry, intercepts and limits at infinity;
- all local and absolute maxima and minima;
- all asymptotes;
- the range.

Then sketch the graph.

Problem 2: basic information and asymptotes

Since $x^2 + 1 > 0$ for every real x ,

$$\text{Dom}(f) = \mathbb{R}.$$

Moreover,

$$f(-x) = -f(x),$$

so the function is odd. Its only intercept is $(0, 0)$.

At infinity,

$$\lim_{x \rightarrow \pm\infty} \frac{x}{x^2 + 1} = 0.$$

$y = 0$ is a horizontal asymptote. There is no vertical asymptote.

Problem 2: derivative and sign chart

By the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The denominator is always positive. The numerator vanishes at $x = \pm 1$.

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	—	+	—
f	decreasing	increasing	decreasing

Problem 2: local and absolute extrema

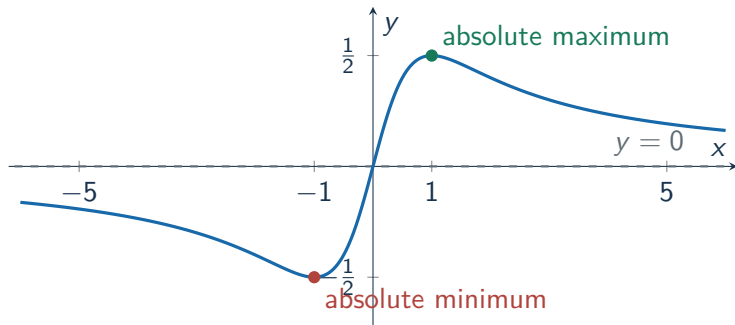
At the critical points,

$$f(-1) = -\frac{1}{2}, \quad f(1) = \frac{1}{2}.$$

The sign chart shows a local minimum at $x = -1$ and a local maximum at $x = 1$. Since the graph approaches 0 at both infinities and the function is monotone on the three intervals, these extrema are also absolute.

$$\min f = -\frac{1}{2} \text{ at } x = -1, \text{ and } \max f = \frac{1}{2} \text{ at } x = 1.$$

Problem 2: complete graph



$$f(\mathbb{R}) = \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Problem 3: vertical and oblique asymptotes

Problem 3

Carry out a complete graph analysis of

$$f(x) = \frac{x^2 + 1}{x - 1}.$$

Find:

- the domain, intercepts and limits at the boundary of the domain;
- all local and absolute maxima and minima;
- the vertical and oblique asymptotes;
- the range.

Then sketch both branches of the graph.

Problem 3: domain and intercepts

The denominator vanishes at $x = 1$, so

$$\text{Dom}(f) = \mathbb{R} \setminus \{1\}.$$

There is no x -intercept because

$$x^2 + 1 > 0$$

for every real x .

At $x = 0$,

$$f(0) = -1.$$

The only intercept is the y -intercept $(0, -1)$.

Problem 3: asymptotes

Polynomial division gives

$$f(x) = x + 1 + \frac{2}{x - 1}.$$

Therefore,

$$\lim_{x \rightarrow \pm\infty} (f(x) - (x + 1)) = 0,$$

so $y = x + 1$ is an oblique asymptote.

At the excluded point,

$$\lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty.$$

$x = 1$ is a vertical asymptote and $y = x + 1$ is an oblique asymptote.

Problem 3: derivative and critical points

Differentiate the divided expression:

$$f'(x) = 1 - \frac{2}{(x-1)^2} = \frac{(x-1)^2 - 2}{(x-1)^2}.$$

The critical points satisfy

$$(x-1)^2 = 2,$$

and hence

$$x = 1 - \sqrt{2} \quad \text{or} \quad x = 1 + \sqrt{2}.$$

The value $x = 1$ is not a critical point because it is not in the domain, but it must separate the two branches in the sign chart.

Problem 3: sign chart and local extrema

x	$(-\infty, 1 - \sqrt{2})$	$(1 - \sqrt{2}, 1)$	$(1, 1 + \sqrt{2})$	$(1 + \sqrt{2}, +\infty)$
$f'(x)$	+	-	-	+
f	increasing	decreasing	decreasing	increasing

At the critical points,

$$f(1 - \sqrt{2}) = 2 - 2\sqrt{2}, \quad f(1 + \sqrt{2}) = 2 + 2\sqrt{2}.$$

$(1 - \sqrt{2}, 2 - 2\sqrt{2})$ is a local maximum and
 $(1 + \sqrt{2}, 2 + 2\sqrt{2})$ is a local minimum.

Problem 3: why the extrema are not absolute

On the left branch,

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow 1^-} f(x) = -\infty.$$

Thus its local maximum is the largest value only on that branch.

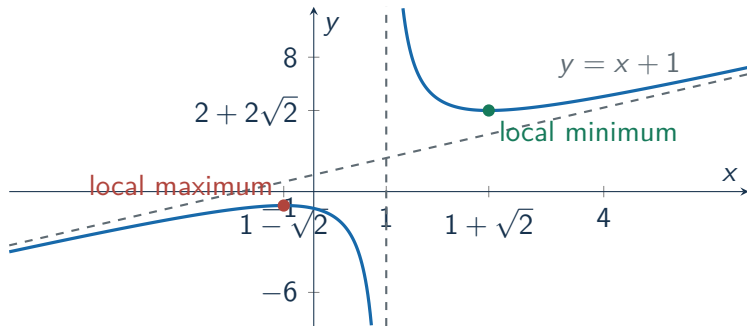
On the right branch,

$$\lim_{x \rightarrow 1^+} f(x) = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

Thus its local minimum is the smallest value only on that branch.

The function has neither an absolute maximum nor an absolute minimum.

Problem 3: complete graph



$$f(\text{Dom } f) = (-\infty, 2 - 2\sqrt{2}] \cup [2 + 2\sqrt{2}, +\infty).$$

Problem 4: a logarithmic function

Problem 4

Carry out a complete graph analysis of

$$f(x) = \frac{\ln x}{x}.$$

Find:

- the domain, intercepts and limiting behaviour;
- all local and absolute maxima and minima;
- all asymptotes;
- the range.

Then sketch the graph.

Problem 4: domain, intercepts and limits

The logarithm requires $x > 0$, so

$$\text{Dom}(f) = (0, +\infty).$$

The graph has no y -intercept. Its only x -intercept satisfies

$$\ln x = 0,$$

and hence is $(1, 0)$.

At the ends of the domain,

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty, \quad \lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0.$$

$x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote.

Problem 4: derivative and monotonicity

Using the quotient rule,

$$f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

Since $x^2 > 0$, the sign depends only on $1 - \ln x$. The unique critical point is

$$\ln x = 1 \iff x = e.$$

x	$(0, e)$	$(e, +\infty)$
$f'(x)$	+	-
f	increasing	decreasing

Problem 4: extrema and range

At the critical point,

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}.$$

The function increases up to this value and decreases afterwards. Hence, $(e, 1/e)$ is both a local maximum and the absolute maximum.

However,

$$f(x) \longrightarrow -\infty \quad \text{as } x \rightarrow 0^+,$$

so there is no absolute minimum.

$$\max f = \frac{1}{e} \text{ at } x = e, \text{ and } f((0, +\infty)) = (-\infty, 1/e].$$

Problem 4: complete graph

