

Calculus 1

Week 5

Problem 1. Find the local and absolute extrema of

$$f(x) = x^3 - 3x^2 - 9x + 5$$

on the interval $[-2, 5]$.

Solution. Since f is continuous on the closed bounded interval $[-2, 5]$, the Extreme Value Theorem guarantees that it attains an absolute maximum and an absolute minimum. Differentiate:

$$f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3).$$

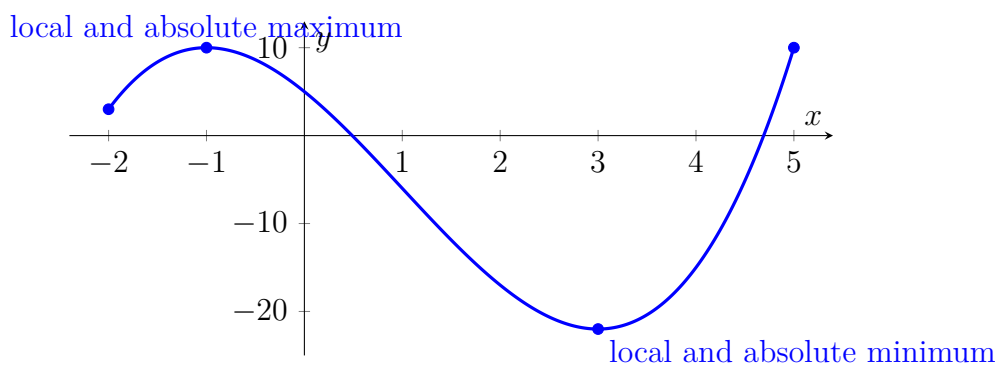
The critical points in the interval are $x = -1$ and $x = 3$. The derivative has the following signs:

x	$(-2, -1)$	$(-1, 3)$	$(3, 5)$
$f'(x)$	+	-	+
f	increasing	decreasing	increasing

Thus, $x = -1$ is a local maximum and $x = 3$ is a local minimum. For the absolute extrema, compare the values at the critical points and the endpoints:

$$f(-2) = 3, \quad f(-1) = 10, \quad f(3) = -22, \quad f(5) = 10.$$

Consequently, the absolute maximum is 10, attained at both $x = -1$ and $x = 5$, and the absolute minimum is -22 , attained at $x = 3$.



Problem 2. Carry out a complete graph analysis of

$$f(x) = x^4 - 4x^2 + 3.$$

Determine its intercepts, intervals of monotonicity, local and absolute extrema, range and asymptotes, and then sketch its graph.

Solution. The domain is $\mathbb{R}$. Since

$$f(-x) = f(x),$$

the function is even and its graph is symmetric with respect to the y -axis. Factoring gives

$$f(x) = (x^2 - 1)(x^2 - 3).$$

Hence, the x -intercepts are

$$x = -\sqrt{3}, -1, 1, \sqrt{3},$$

and the y -intercept is $(0, 3)$. Moreover,

$$\lim_{x \rightarrow -\infty} f(x) = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

Since f is a polynomial, it has no vertical, horizontal or oblique asymptote.

Differentiate and factor:

$$f'(x) = 4x^3 - 8x = 4x(x - \sqrt{2})(x + \sqrt{2}).$$

The critical points are $-\sqrt{2}$, 0 and $\sqrt{2}$. The sign chart is

x	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, +\infty)$
$f'(x)$	—	+	—	+
f	decreasing	increasing	decreasing	increasing

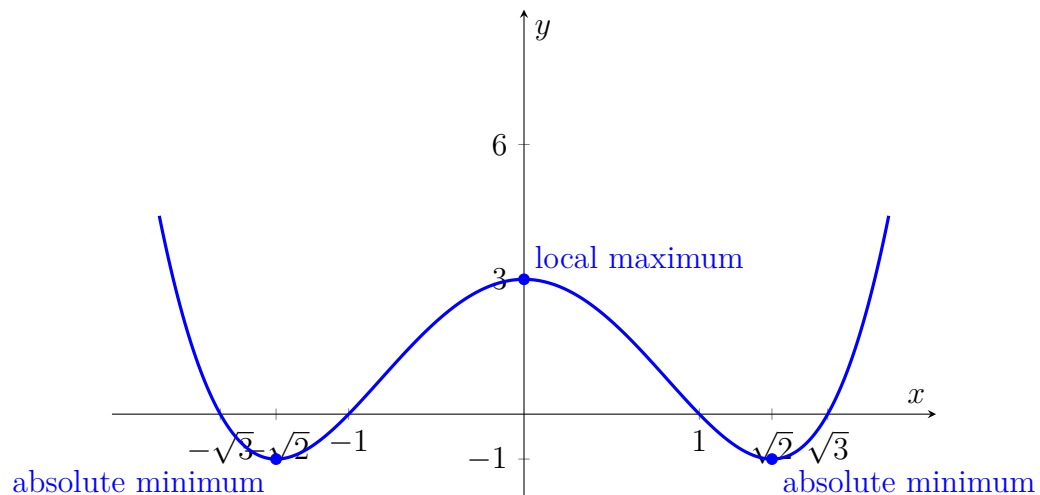
The values at the critical points are

$$f(-\sqrt{2}) = -1, \quad f(0) = 3, \quad f(\sqrt{2}) = -1.$$

Thus, $(0, 3)$ is a local maximum. The points $(\pm\sqrt{2}, -1)$ are local minima. Because $f(x) \rightarrow +\infty$ at both infinities, these two points are also absolute minima, while the function has no absolute maximum. Therefore,

$$f(\mathbb{R}) = [-1, +\infty).$$

The zeros, extrema, symmetry and end behaviour determine the following graph.



Problem 3. Carry out a complete graph analysis of

$$f(x) = \frac{x}{x^2 + 1}.$$

Determine the asymptotes, local and absolute extrema and range.

Solution. Since $x^2 + 1 > 0$ for every $x \in \mathbb{R}$, the domain is $\mathbb{R}$. Also,

$$f(-x) = -f(x),$$

so the function is odd. Its only intercept is $(0, 0)$. At infinity,

$$\lim_{x \rightarrow -\infty} \frac{x}{x^2 + 1} = 0, \quad \lim_{x \rightarrow +\infty} \frac{x}{x^2 + 1} = 0.$$

Hence, $y = 0$ is a horizontal asymptote. There is no vertical asymptote because the denominator never vanishes.

By the quotient rule,

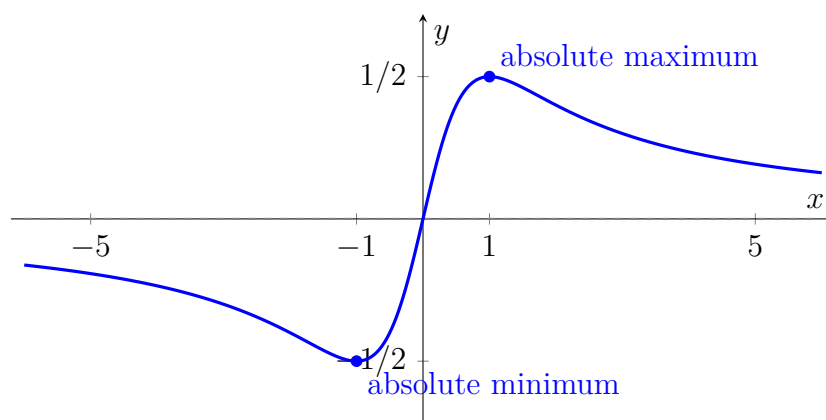
$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The denominator is positive, so the sign is determined by $1 - x^2$. Therefore,

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	—	+	—
f	decreasing	increasing	decreasing

The point $(-1, -1/2)$ is a local minimum, and $(1, 1/2)$ is a local maximum. The limits at infinity and the monotonicity on the three intervals show that they are also the absolute minimum and the absolute maximum. Consequently,

$$f(\mathbb{R}) = \left[-\frac{1}{2}, \frac{1}{2}\right].$$



Problem 4. Carry out a complete graph analysis of

$$f(x) = \frac{x^2 + 1}{x - 1}.$$

Determine its intercepts, asymptotes, monotonicity, local and absolute extrema and range.

Solution. The denominator vanishes at $x = 1$, so

$$\text{Dom}(f) = \mathbb{R} \setminus \{1\}.$$

There is no x -intercept because $x^2 + 1 > 0$ for every real x . The y -intercept is $(0, -1)$.

Polynomial division gives

$$f(x) = x + 1 + \frac{2}{x - 1}.$$

Therefore,

$$\lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty,$$

and $x = 1$ is a vertical asymptote. Moreover,

$$\lim_{x \rightarrow \pm\infty} (f(x) - (x + 1)) = 0,$$

so $y = x + 1$ is an oblique asymptote.

Differentiating the divided expression,

$$f'(x) = 1 - \frac{2}{(x - 1)^2} = \frac{(x - 1)^2 - 2}{(x - 1)^2}.$$

The critical points are

$$x = 1 - \sqrt{2} \quad \text{and} \quad x = 1 + \sqrt{2}.$$

The point $x = 1$ is not a critical point because it does not belong to the domain, but it separates the two branches of the graph.

The derivative has the following signs:

x	$(-\infty, 1 - \sqrt{2})$	$(1 - \sqrt{2}, 1)$	$(1, 1 + \sqrt{2})$	$(1 + \sqrt{2}, +\infty)$
$f'(x)$	+	-	-	+
f	increasing	decreasing	decreasing	increasing

Since

$$f(1 - \sqrt{2}) = 2 - 2\sqrt{2}, \quad f(1 + \sqrt{2}) = 2 + 2\sqrt{2},$$

the first point is a local maximum and the second is a local minimum. Neither is an absolute extremum on the whole domain: the left branch is unbounded below, while the right branch is unbounded above.

To determine the range, write

$$y = \frac{x^2 + 1}{x - 1}.$$

Then x must satisfy

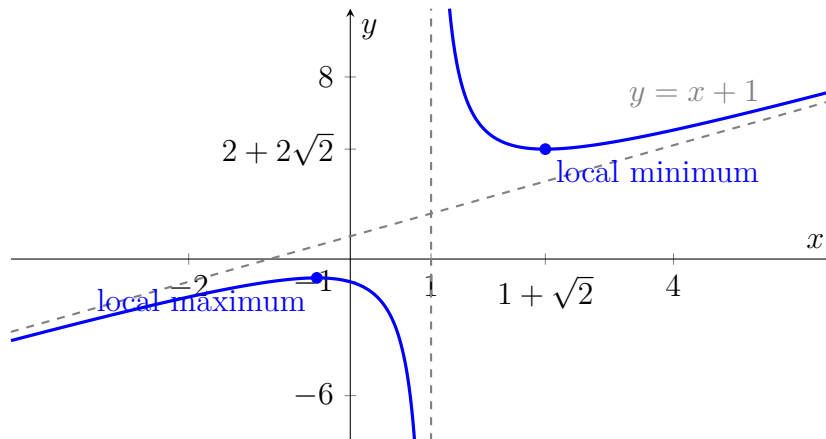
$$x^2 - yx + (y + 1) = 0.$$

This quadratic has a real solution precisely when

$$y^2 - 4y - 4 \geq 0.$$

Thus,

$$f(\text{Dom } f) = (-\infty, 2 - 2\sqrt{2}] \cup [2 + 2\sqrt{2}, +\infty).$$



Problem 5. Carry out a complete graph analysis of

$$f(x) = \frac{\ln x}{x}.$$

Determine its intercepts, asymptotes, monotonicity, local and absolute extrema and range.

Solution. The natural domain is $(0, +\infty)$. The graph has no y -intercept, and its only x -intercept is $(1, 0)$. At the ends of the domain,

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty, \quad \lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0.$$

Hence, $x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote at $+\infty$.

Using the quotient rule,

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Since $x^2 > 0$, the sign depends only on $1 - \ln x$. The unique critical point is $x = e$, and

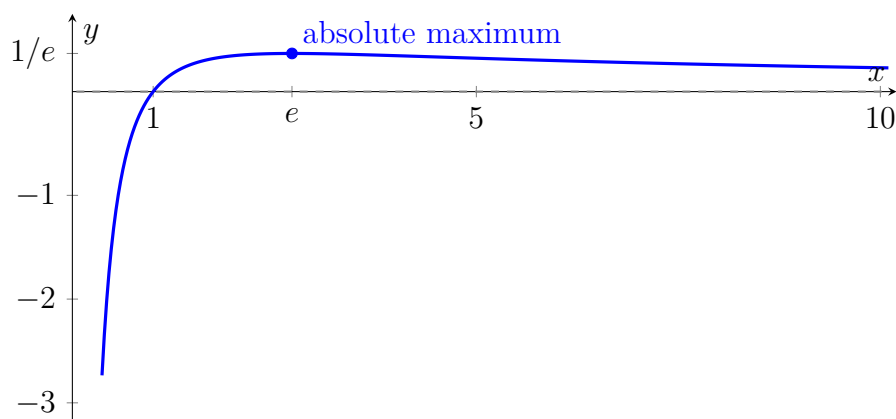
x	$(0, e)$	$(e, +\infty)$
$f'(x)$	+	-
f	increasing	decreasing

Therefore,

$$f(e) = \frac{1}{e}$$

is both a local and the absolute maximum. There is no absolute minimum because the function tends to $-\infty$ as $x \rightarrow 0^+$. Thus,

$$f((0, +\infty)) = (-\infty, 1/e].$$



Problem 6. Let f be the continuous piecewise linear function whose graph joins the points

$$(-4, 5), \quad (-1, 2), \quad (2, 4), \quad (4, 0)$$

in this order.

- (a) Sketch the graph of f' .
- (b) Determine the intervals on which f is increasing or decreasing.
- (c) Find the local and absolute extrema of f .

Solution. On each open segment, the derivative equals the slope of that segment. The slopes are

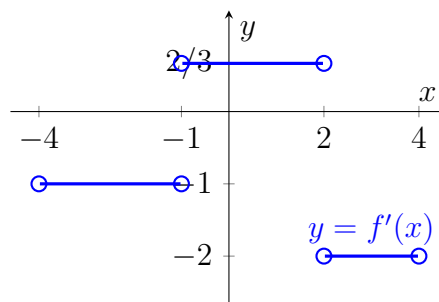
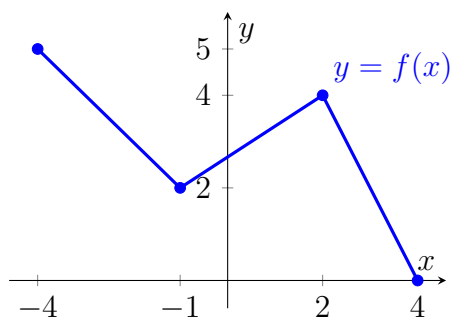
$$\frac{2-5}{-1-(-4)} = -1, \quad \frac{4-2}{2-(-1)} = \frac{2}{3}, \quad \frac{0-4}{4-2} = -2.$$

Consequently,

$$f'(x) = \begin{cases} -1, & -4 < x < -1, \\ \frac{2}{3}, & -1 < x < 2, \\ -2, & 2 < x < 4. \end{cases}$$

The derivative does not exist at the corners $x = -1$ and $x = 2$. With the usual two-sided definition, it is also not defined at the endpoints.

The function is decreasing on $(-4, -1)$, increasing on $(-1, 2)$ and decreasing on $(2, 4)$. Thus, $(-1, 2)$ is an interior local minimum and $(2, 4)$ is an interior local maximum. Comparing all vertex values, the absolute maximum is 5 at $x = -4$, and the absolute minimum is 0 at $x = 4$.



Problem 7. Carry out a complete graph analysis of

$$f(x) = x + \frac{4}{x}.$$

Determine its intercepts, asymptotes, monotonicity, local and absolute extrema and range.

Solution. The domain is $\mathbb{R} \setminus \{0\}$. There is no y -intercept. An x -intercept would have to satisfy

$$x + \frac{4}{x} = 0 \iff x^2 + 4 = 0,$$

which has no real solution. Thus, the graph has no intercepts.

At the excluded point,

$$\lim_{x \rightarrow 0^-} f(x) = -\infty, \quad \lim_{x \rightarrow 0^+} f(x) = +\infty.$$

Hence, $x = 0$ is a vertical asymptote. Moreover,

$$\lim_{x \rightarrow \pm\infty} (f(x) - x) = \lim_{x \rightarrow \pm\infty} \frac{4}{x} = 0,$$

so $y = x$ is an oblique asymptote.

Differentiating,

$$f'(x) = 1 - \frac{4}{x^2} = \frac{(x-2)(x+2)}{x^2}.$$

The critical points are $x = -2$ and $x = 2$. The excluded point $x = 0$ also separates the two branches and must appear in the sign chart.

We obtain

x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	+	-	-	+
f	increasing	decreasing	decreasing	increasing

Since

$$f(-2) = -4, \quad f(2) = 4,$$

the point $(-2, -4)$ is a local maximum and $(2, 4)$ is a local minimum. Neither is an absolute extremum on the whole domain because the function is unbounded above and below.

To determine the range, set $y = x + 4/x$. Then

$$x^2 - yx + 4 = 0.$$

This equation has a real solution if and only if

$$y^2 - 16 \geq 0.$$

Therefore,

$$f(\mathbb{R} \setminus \{0\}) = (-\infty, -4] \cup [4, +\infty).$$

