

Calculus 1

Week 5 (extra): Graph Analysis

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One method for every graph

For each function, we proceed in the same order:

1. determine the domain, symmetry and where the graph meets the coordinate axes;
2. calculate boundary limits and identify asymptotes;
3. use the sign of f' to locate and classify every extremum;
4. use the sign of f'' to study concavity and inflection points;
5. determine the range and draw the graph from the analysis.

Problem 1

Problem 1

Carry out a complete graph analysis of

$$f(x) = xe^{-|x|}.$$

Find the maximal intervals of increase and decrease, all local and absolute extrema, the concavity intervals, the inflection points and the range. Identify all asymptotes and sketch the graph.

Problem 1

Write the function piecewise:

$$f(x) = \begin{cases} xe^x, & x < 0, \\ xe^{-x}, & x \geq 0. \end{cases}$$

Its domain is $\mathbb{R}$, and

$$f(-x) = -f(x),$$

so f is odd. The graph meets both coordinate axes only at $(0, 0)$.

Moreover,

$$\lim_{x \rightarrow -\infty} f(x) = 0^-, \quad \lim_{x \rightarrow +\infty} f(x) = 0^+.$$

$y = 0$ is a horizontal asymptote at both ends.

Problem 1

The function is differentiable at 0, with $f'(0) = 1$, and

$$f'(x) = \begin{cases} e^x(x+1), & x < 0, \\ e^{-x}(1-x), & x > 0. \end{cases}$$

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
$f'(x)$	—	+	+	—
f	decreasing	increasing	increasing	decreasing

$(-1, -1/e)$ is the absolute minimum and $(1, 1/e)$ is the absolute maximum. The maximal increasing interval is $(-1, 1)$.

Problem 1

For $x \neq 0$,

$$f''(x) = \begin{cases} e^x(x+2), & x < 0, \\ e^{-x}(x-2), & x > 0. \end{cases}$$

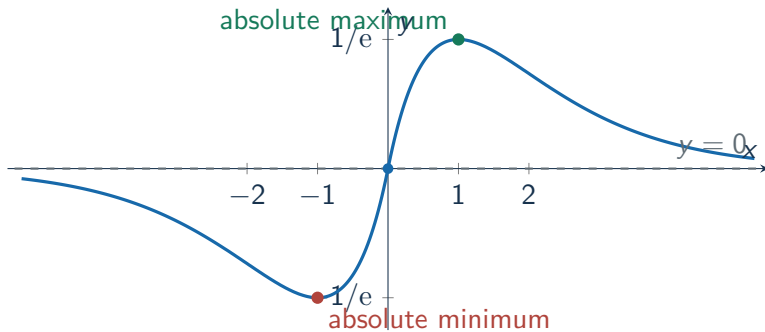
x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, +\infty)$
$f''(x)$	—	+	—	+
f	concave down	concave up	concave down	concave up

The concavity changes at all three separating points. Thus the inflection points are

$$\left(-2, -\frac{2}{e^2}\right), \quad (0, 0), \quad \left(2, \frac{2}{e^2}\right).$$

The second derivative need not exist at an inflection point.

Problem 1



$$f(\mathbb{R}) = [-1/e, 1/e].$$

Problem 2

Problem 2

Carry out a complete graph analysis of

$$f(x) = \arctan\left(\left|x^2 - 6x\right|\right).$$

Find all extrema and the maximal intervals of increase and decrease. Determine the asymptotes and the range, and then sketch the graph.

Problem 2

The expression inside the absolute value changes sign at 0 and 6. Hence

$$f(x) = \begin{cases} \arctan(x^2 - 6x), & x \leq 0 \text{ or } x \geq 6, \\ \arctan(6x - x^2), & 0 < x < 6. \end{cases}$$

The domain is $\mathbb{R}$. The graph meets the x -axis at $(0,0)$ and $(6,0)$, and it meets the y -axis at $(0,0)$. Finally,

$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{\pi}{2}.$$

$y = \pi/2$ is a horizontal asymptote; it is never attained.

Problem 2

Differentiate the two pieces:

$$f'(x) = \begin{cases} \frac{2x - 6}{1 + (x^2 - 6x)^2}, & x < 0 \text{ or } x > 6, \\ \frac{6 - 2x}{1 + (6x - x^2)^2}, & 0 < x < 6. \end{cases}$$

The derivative does not exist at 0 or 6.

x	$(-\infty, 0)$	$(0, 3)$	$(3, 6)$	$(6, +\infty)$
$f'(x)$	—	+	—	+
f	decreasing	increasing	decreasing	increasing

Problem 2

At the three relevant values,

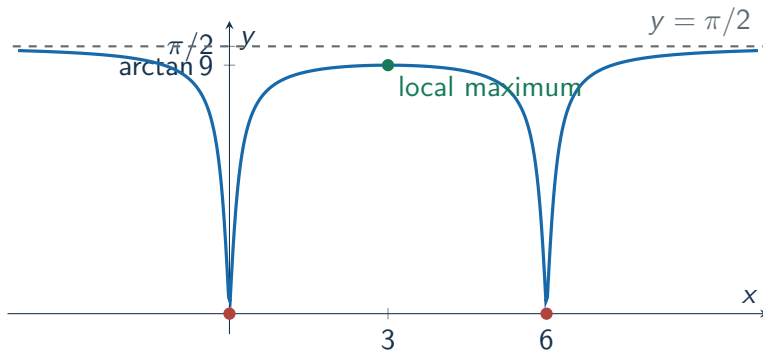
$$f(0) = f(6) = 0, \quad f(3) = \arctan 9.$$

Therefore:

- $(0, 0)$ and $(6, 0)$ are local and absolute minima;
- $(3, \arctan 9)$ is a local maximum;
- there is no absolute maximum, because $f(x) < \pi/2$ for every x .

$$f(\mathbb{R}) = [0, \pi/2).$$

Problem 2



The absolute minima occur at the two cusps $x = 0$ and $x = 6$.

Problem 3

Problem 3

Carry out a complete graph analysis of

$$f(x) = \sqrt{x^4 + 2x^2}.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph.

Problem 3

The essential simplification is

$$f(x) = \sqrt{x^2(x^2 + 2)} = |x| \sqrt{x^2 + 2}.$$

Therefore, the domain is $\mathbb{R}$, f is even, and the graph meets both coordinate axes only at $(0, 0)$. Moreover,

$$\lim_{x \rightarrow \pm\infty} f(x) = +\infty.$$

There are no vertical, horizontal or oblique linear asymptotes. However,

$$f(x) - (x^2 + 1) = -\frac{1}{\sqrt{x^4 + 2x^2} + x^2 + 1} \rightarrow 0.$$

The curve approaches the parabola $y = x^2 + 1$ at both ends.

Problem 3

For $x \neq 0$,

$$f'(x) = \begin{cases} -\frac{2(x^2 + 1)}{\sqrt{x^2 + 2}}, & x < 0, \\ \frac{2(x^2 + 1)}{\sqrt{x^2 + 2}}, & x > 0. \end{cases}$$

At 0 the one-sided derivatives are

$$f'_-(0) = -\sqrt{2}, \quad f'_+(0) = \sqrt{2}.$$

Thus the graph has a cusp at the origin.

f decreases on $(-\infty, 0)$ and increases on $(0, +\infty)$. The point $(0, 0)$ is the local and absolute minimum; there is no maximum.

Problem 3

For $x \neq 0$,

$$f''(x) = \frac{2|x|(x^2 + 3)}{(x^2 + 2)^{3/2}} > 0.$$

Hence the graph is concave up on both sides of the origin. In fact, it is convex on all of $\mathbb{R}$.

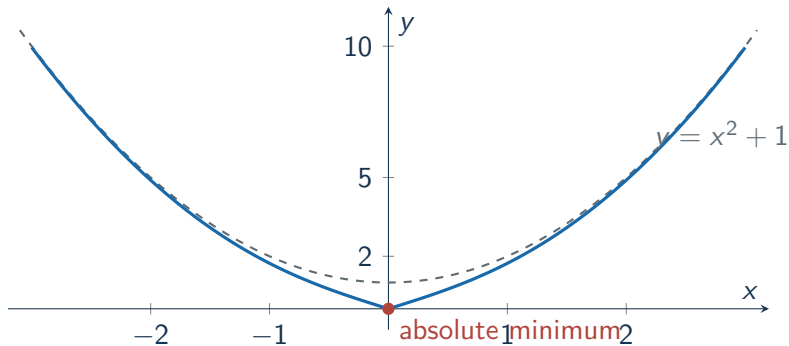
Although f' does not exist at 0, the concavity does not change there. Thus there are no inflection points.

Useful points for the sketch are

$$(0, 0), \quad (\pm 1, \sqrt{3}), \quad (\pm \sqrt{2}, 2\sqrt{2}).$$

$$f(\mathbb{R}) = [0, +\infty).$$

Problem 3



The cusp comes from $\sqrt{x^2} = |x|$, not from x .

Problem 4

Problem 4

On the interval $[-1, 6]$, carry out a complete graph analysis of

$$f(x) = x^2 - 8|x - 1| + 8.$$

In particular, find the absolute maximum and minimum, all interior local extrema, the concavity intervals and the range. Then sketch the restricted graph.

Problem 4

On the stated domain,

$$f(x) = \begin{cases} x^2 + 8x, & -1 \leq x < 1, \\ (x - 4)^2, & 1 \leq x \leq 6. \end{cases}$$

Both formulas give $f(1) = 9$, so the function is continuous at the junction.

The graph meets the x -axis at $(0, 0)$ and $(4, 0)$. Its intersection with the y -axis is $(0, 0)$.

The relevant endpoint and junction values are

$$f(-1) = -7, \quad f(1) = 9, \quad f(6) = 4.$$

There are no asymptotes because the domain is bounded.

Problem 4

For interior points different from 1,

$$f'(x) = \begin{cases} 2x + 8, & -1 < x < 1, \\ 2x - 8, & 1 < x < 6. \end{cases}$$

At the junction,

$$f'_-(1) = 10, \quad f'_+(1) = -6,$$

so f' does not exist at 1.

x	$(-1, 1)$	$(1, 4)$	$(4, 6)$
$f'(x)$	+	-	+
f	increasing	decreasing	increasing

Problem 4

We compare all candidates:

x	-1	1	4	6
$f(x)$	-7	9	0	4

Therefore:

- $(1, 9)$ is an interior local maximum and the absolute maximum;
- $(4, 0)$ is an interior local minimum, but not the absolute minimum;
- $(-1, -7)$ is the absolute minimum.

If one-sided local extrema are counted relative to the restricted domain, the left endpoint is also a local minimum and the right endpoint is also a local maximum.

$$f([-1, 6]) = [-7, 9].$$

Problem 4

On both smooth pieces,

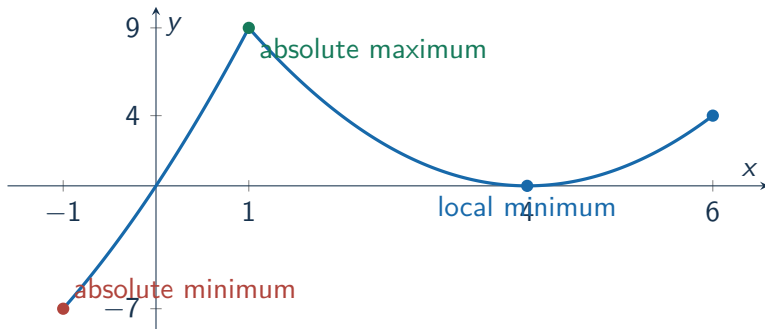
$$f''(x) = 2 > 0.$$

Thus the graph is concave up on $(-1, 1)$ and on $(1, 6)$.

At $x = 1$ the derivative jumps from 10 to -6 , but the concavity is upward on both sides. Hence $(1, 9)$ is a sharp local maximum, not an inflection point.

The restricted graph has no inflection points.

Problem 4



Problem 5

Problem 5

Carry out a complete graph analysis of

$$f(x) = x^3 - |x|^3 + 3x^2 - 12x.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph.

Problem 5

Removing the absolute value gives

$$f(x) = \begin{cases} 2x^3 + 3x^2 - 12x, & x < 0, \\ 3x^2 - 12x, & x \geq 0. \end{cases}$$

The domain is $\mathbb{R}$ and the function is neither even nor odd.

The graph meets the x -axis at

$$\left(\frac{-3 - \sqrt{105}}{4}, 0 \right), \quad (0, 0), \quad (4, 0).$$

Its intersection with the y -axis is $(0, 0)$. The other root of $2x^2 + 3x - 12$ is positive and does not belong to the branch $x < 0$.

Finally,

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

There are no linear asymptotes.

Problem 5

The two first-derivative formulas agree at 0, where $f'(0) = -12$:

$$f'(x) = \begin{cases} 6(x+2)(x-1), & x < 0, \\ 6(x-2), & x > 0. \end{cases}$$

x	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, +\infty)$
$f'(x)$	+	-	-	+
f	increasing	decreasing	decreasing	increasing

$(-2, 20)$ is a local maximum and $(2, -12)$ is a local minimum.
Neither is absolute because f is unbounded above and below.

Problem 5

The second derivatives also agree at 0:

$$f''(x) = \begin{cases} 12x + 6, & x < 0, \\ 6, & x > 0, \end{cases} \quad f''(0) = 6.$$

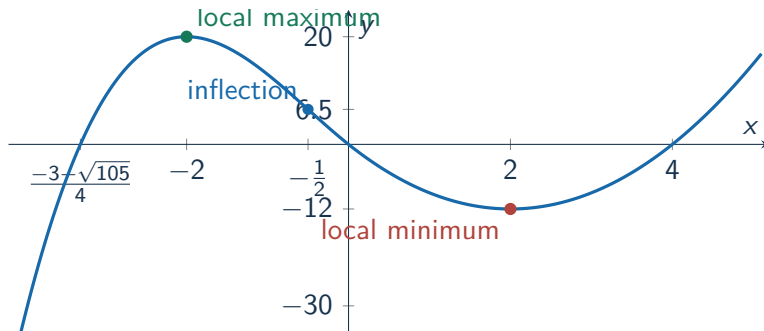
Therefore, f is concave down on $(-\infty, -1/2)$ and concave up on $(-1/2, +\infty)$. The unique inflection point is

$$\left(-\frac{1}{2}, \frac{13}{2}\right).$$

Since f is continuous and its end limits are $-\infty$ and $+\infty$,

$$f(\mathbb{R}) = \mathbb{R}.$$

Problem 5



Problem 6

Problem 6

Carry out a complete graph analysis of

$$f(x) = 3x^4 - 8x^3 + 6x^2.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph.

Problem 6

Factor the polynomial:

$$f(x) = x^2(3x^2 - 8x + 6).$$

The discriminant of the quadratic factor is

$$(-8)^2 - 4 \cdot 3 \cdot 6 = -8 < 0,$$

and its leading coefficient is positive. Thus the graph meets both coordinate axes only at $(0, 0)$. As a zero of f , $x = 0$ has multiplicity two.

The domain is $\mathbb{R}$, the function has no symmetry, and

$$\lim_{x \rightarrow \pm\infty} f(x) = +\infty.$$

There are no asymptotes.

Problem 6

Differentiate and factor:

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x - 1)^2.$$

x	$(-\infty, 0)$	$(0, 1)$	$x = 1$	$(1, +\infty)$
$f'(x)$	—	+	0	+
f	decreasing	increasing	stationary	increasing

The derivative does not change sign at $x = 1$, so $(1, 1)$ is not an extremum.

$(0, 0)$ is the local and absolute minimum. There is no maximum, and $f(\mathbb{R}) = [0, +\infty)$.

Problem 6

$$f''(x) = 36x^2 - 48x + 12 = 12(3x - 1)(x - 1).$$

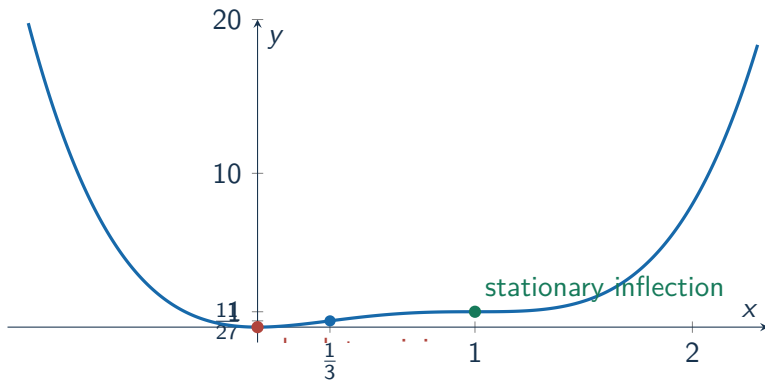
x	$(-\infty, 1/3)$	$(1/3, 1)$	$(1, +\infty)$
$f''(x)$	+	-	+
f	concave up	concave down	concave up

Hence the inflection points are

$$\left(\frac{1}{3}, \frac{11}{27}\right), \quad (1, 1).$$

The point $(1, 1)$ is a **stationary inflection point**: $f'(1) = 0$, but it is not a local maximum or minimum.

Problem 6



Problem 7

Problem 7

Carry out a complete graph analysis of

$$f(x) = e^{x^2+3x-|x|}.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph.

Problem 7

The exponent is

$$x^2 + 3x - |x| = \begin{cases} x^2 + 4x = (x + 2)^2 - 4, & x < 0, \\ x^2 + 2x = (x + 1)^2 - 1, & x \geq 0. \end{cases}$$

The domain is $\mathbb{R}$. The function is positive, so the graph does not meet the x -axis. It meets the y -axis at $(0, 1)$. It is neither even nor odd, and

$$\lim_{x \rightarrow \pm\infty} f(x) = +\infty.$$

There are no asymptotes.

Problem 7

For $x \neq 0$,

$$f'(x) = \begin{cases} 2(x+2)e^{x^2+4x}, & x < 0, \\ 2(x+1)e^{x^2+2x}, & x > 0. \end{cases}$$

At the origin,

$$f'_-(0) = 4, \quad f'_+(0) = 2,$$

so the function is continuous but not differentiable there.

x	$(-\infty, -2)$	$(-2, 0)$	$(0, +\infty)$
$f'(x)$	—	+	+
f	decreasing	increasing	increasing

Problem 7

At the sign change,

$$f(-2) = e^{-4}.$$

Thus $(-2, e^{-4})$ is the local and absolute minimum. The corner at $x = 0$ is a critical number but not an extremum. There is no maximum.

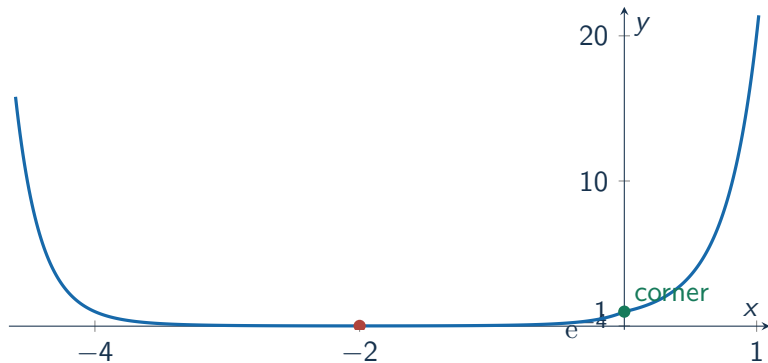
For $x \neq 0$,

$$f''(x) = \begin{cases} (4(x+2)^2 + 2)e^{x^2+4x}, & x < 0, \\ (4(x+1)^2 + 2)e^{x^2+2x}, & x > 0. \end{cases}$$

Both expressions are positive. The graph is concave up on $(-\infty, 0)$ and $(0, +\infty)$; there is no change of concavity and no inflection point.

$$f(\mathbb{R}) = [e^{-4}, +\infty).$$

Problem 7



Problem 8

Problem 8

Carry out a complete graph analysis of

$$f(x) = \frac{\ln(x+1)}{x}.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph, including every excluded point.

Problem 8

The natural domain is

$$(-1, 0) \cup (0, +\infty).$$

The graph meets neither coordinate axis: the numerator vanishes only at $x = 0$, and this value is excluded from the domain.

At the boundaries of the domain,

$$\lim_{x \rightarrow -1^+} f(x) = +\infty, \quad \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 1, \quad \lim_{x \rightarrow +\infty} f(x) = 0.$$

$x = -1$ is a vertical asymptote, $y = 0$ is a horizontal asymptote, and $(0, 1)$ is a removable hole.

Problem 8

By the quotient rule, let

$$f'(x) = \frac{N(x)}{x^2}, \quad N(x) = \frac{x}{x+1} - \ln(x+1).$$

Its continuous extension satisfies $N(0) = 0$, and

$$N'(x) = -\frac{x}{(x+1)^2}.$$

Thus N increases up to 0 and decreases after 0, so

$$N(x) < 0 \quad (x \neq 0).$$

$f'(x) < 0$ on both $(-1, 0)$ and $(0, +\infty)$. There are no local or absolute extrema.

Problem 8

Differentiate once more. We obtain

$$f''(x) = \frac{H(x)}{x^3}, \quad H(x) = 2 \ln(x+1) - \frac{2x}{x+1} - \frac{x^2}{(x+1)^2}.$$

Now

$$H'(x) = \frac{2x^2}{(x+1)^3} \geq 0, \quad H(0) = 0.$$

Therefore $H(x)$ and x^3 have the same sign whenever $x \neq 0$. Hence $f''(x) > 0$ throughout the domain.

Both branches are concave up. There is no inflection point because the only possible separating point, $x = 0$, is not on the graph.

Problem 8

On the left branch, the function decreases from $+\infty$ to the missing value 1:

$$f((-1, 0)) = (1, +\infty).$$

On the right branch, it decreases from the missing value 1 to the unattained horizontal asymptote 0:

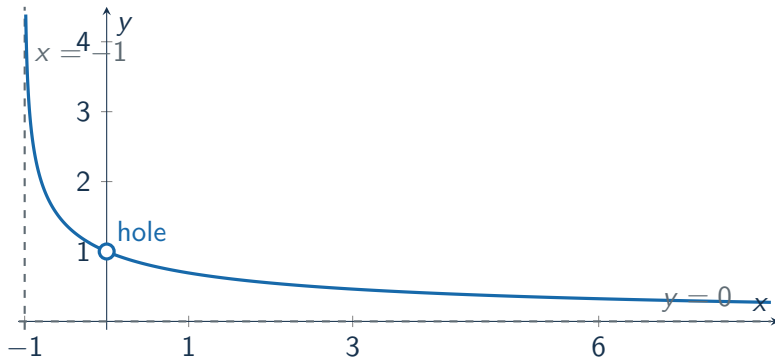
$$f((0, +\infty)) = (0, 1).$$

Consequently,

$$f(\text{Dom } f) = (0, 1) \cup (1, +\infty).$$

The missing point $(0, 1)$ removes the value 1 from the range.

Problem 8



Problem 9

Problem 9

On the stated interval $[-1, 6]$, analyze

$$f(x) = \frac{x^2 + x - 2}{x^2}.$$

Find the absolute maximum and minimum if they exist, complete the monotonicity and concavity analysis, determine the range and sketch the graph.

Problem 9

The natural domain inside the stated interval is

$$D = [-1, 0) \cup (0, 6].$$

Although $[-1, 6]$ is closed and bounded, f is not defined at 0. Therefore, the Extreme Value Theorem does **not** apply.

Rewrite

$$f(x) = 1 + \frac{1}{x} - \frac{2}{x^2} = \frac{(x+2)(x-1)}{x^2}.$$

The graph meets the x -axis at $(1, 0)$ but not the y -axis, since $x = 0$ is excluded from the domain. At the missing point,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = -\infty.$$

$x = 0$ is a vertical asymptote.

Problem 9

$$f'(x) = -\frac{1}{x^2} + \frac{4}{x^3} = \frac{4-x}{x^3}.$$

x	$[-1, 0)$	$(0, 4)$	$(4, 6]$
$f'(x)$	—	+	—
f	decreasing	increasing	decreasing

The candidate values are

$$f(-1) = -2, \quad f(4) = \frac{9}{8}, \quad f(6) = \frac{10}{9}.$$

$(4, 9/8)$ is the interior local maximum and the absolute maximum. There is no absolute minimum because $f(x) \rightarrow -\infty$ near 0.

Problem 9

$$f''(x) = \frac{2(x-6)}{x^4}.$$

For every interior point of D , we have $x < 6$, so

$$f''(x) < 0.$$

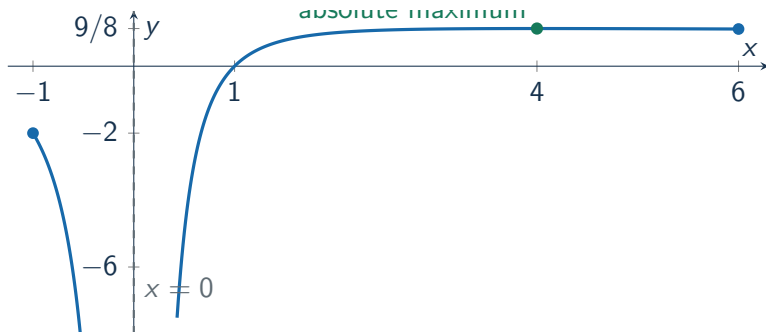
Both branches are concave down. The value $x = 6$ is an endpoint, and there is no change of concavity there. Thus the restricted graph has no inflection points.

The left branch has range $(-\infty, -2]$, while the right branch already covers every value from $-\infty$ to $9/8$. Hence

$$f(D) = (-\infty, 9/8].$$

A bounded interval alone does not guarantee absolute extrema; the domain and continuity hypotheses matter.

Problem 9



Problem 10

Problem 10

Carry out a complete graph analysis of

$$f(x) = \frac{x}{\ln x}.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch both branches of the graph.

Problem 10

The natural domain is

$$(0, 1) \cup (1, +\infty).$$

The graph meets neither coordinate axis: $x = 0$ is excluded and $x / \ln x \neq 0$ throughout the domain.

The relevant limits are

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x}{\ln x} &= 0^-, & \lim_{x \rightarrow 1^-} \frac{x}{\ln x} &= -\infty, \\ \lim_{x \rightarrow 1^+} \frac{x}{\ln x} &= +\infty, & \lim_{x \rightarrow +\infty} \frac{x}{\ln x} &= +\infty.\end{aligned}$$

$x = 1$ is the only vertical asymptote. There is no horizontal or oblique asymptote at infinity.

Problem 10

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}.$$

The denominator is positive on the domain.

x	$(0, 1)$	$(1, e)$	$(e, +\infty)$
$f'(x)$	—	—	+
f	decreasing	decreasing	increasing

At the critical point, $f(e) = e$.

(e, e) is a local minimum and the minimum of the right branch. On the full domain there is neither an absolute maximum nor an absolute minimum.

Problem 10

Differentiate again:

$$f''(x) = \frac{2 - \ln x}{x(\ln x)^3}.$$

x	$(0, 1)$	$(1, e^2)$	$(e^2, +\infty)$
$f''(x)$	—	+	—
f	concave down	concave up	concave down

The concavity changes at $x = e^2$, and

$$f(e^2) = \frac{e^2}{2}.$$

$(e^2, e^2/2)$ is the unique inflection point.

Problem 10

On $(0, 1)$, the function decreases from the missing endpoint value 0 to $-\infty$. Therefore,

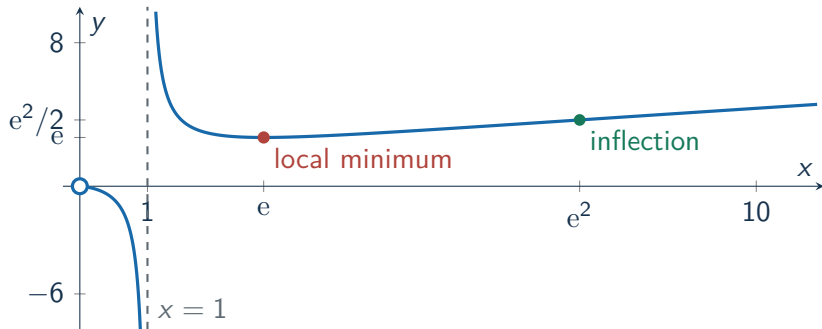
$$f((0, 1)) = (-\infty, 0).$$

On $(1, +\infty)$, the function decreases from $+\infty$ to e and then increases to $+\infty$. Therefore,

$$f((1, +\infty)) = [e, +\infty).$$

$$f(\text{Dom } f) = (-\infty, 0) \cup [e, +\infty).$$

Problem 10



Problem 11

Problem 11

Carry out a complete graph analysis of

$$f(x) = x \ln^2 x.$$

Find all extrema, monotonicity and concavity intervals, inflection points, asymptotes and the range. Then sketch the graph.

Problem 11

The natural domain is $(0, +\infty)$. The graph meets the x -axis only at $(1, 0)$. It does not meet the y -axis because $x = 0$ is outside the domain.

At the ends of the domain,

$$\lim_{x \rightarrow 0^+} x \ln^2 x = 0^+, \quad \lim_{x \rightarrow +\infty} x \ln^2 x = +\infty.$$

Thus the graph approaches the excluded endpoint $(0, 0)$, but $x = 0$ is not a vertical asymptote. There is no horizontal or oblique asymptote at infinity.

Problem 11

$$f'(x) = \ln^2 x + 2 \ln x = \ln x (\ln x + 2).$$

The critical points are $x = e^{-2}$ and $x = 1$.

x	$(0, e^{-2})$	$(e^{-2}, 1)$	$(1, +\infty)$
$f'(x)$	+	-	+
f	increasing	decreasing	increasing

$$f(e^{-2}) = \frac{4}{e^2}, \quad f(1) = 0.$$

$(e^{-2}, 4/e^2)$ is a local maximum. $(1, 0)$ is the local and absolute minimum.
There is no absolute maximum.

Problem 11

$$f''(x) = \frac{2(\ln x + 1)}{x}.$$

Since $x > 0$, the sign changes at $x = e^{-1}$:

x	$(0, e^{-1})$	$(e^{-1}, +\infty)$
$f''(x)$	−	+
f	concave down	concave up

Thus the unique inflection point is

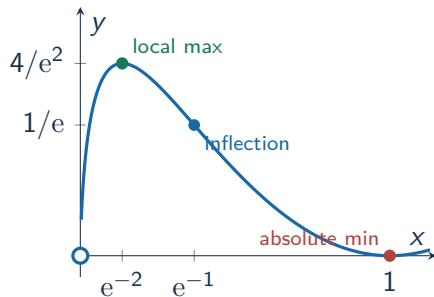
$$(e^{-1}, e^{-1}).$$

Because $f(x) \geq 0$, the value 0 is attained, and the function is unbounded above,

$$f((0, +\infty)) = [0, +\infty).$$

Problem 11

Zoom near the extrema



End behaviour

