

# Calculus 1

## Lecture 3: Limits

Sheldon Dantas

Faculty of Electrical Engineering, Czech Technical University in Prague

2026–2027 · 1st semester

---

# What we will be able to do today

---

- ★ Explain what a limit means using  $\varepsilon$  and  $\delta$ .

# What we will be able to do today

---

- ★ Explain what a limit means using  $\varepsilon$  and  $\delta$ .
- ★ Calculate limits using algebra, comparison and standard limits.

# What we will be able to do today

---

- ★ Explain what a limit means using  $\varepsilon$  and  $\delta$ .
- ★ Calculate limits using algebra, comparison and standard limits.
- ★ Calculate one-sided and infinite limits.

# What we will be able to do today

---

- ★ Explain what a limit means using  $\varepsilon$  and  $\delta$ .
- ★ Calculate limits using algebra, comparison and standard limits.
- ★ Calculate one-sided and infinite limits.
- ★ Recognize and remove basic indeterminate forms.

# What we will be able to do today

---

- ★ Explain what a limit means using  $\varepsilon$  and  $\delta$ .
- ★ Calculate limits using algebra, comparison and standard limits.
- ★ Calculate one-sided and infinite limits.
- ★ Recognize and remove basic indeterminate forms.
- ★ Important theorems.

What a limit measures

# A limit describes behaviour near a point

---

Consider

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

# A limit describes behaviour near a point

---

Consider

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

- The formula is not defined at  $x = 1$ .

# A limit describes behaviour near a point

---

Consider

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

- The formula is not defined at  $x = 1$ .
- We can still ask what happens when  $x$  is close to 1.

# A limit describes behaviour near a point

---

Consider

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

- The formula is not defined at  $x = 1$ .
- We can still ask what happens when  $x$  is close to 1.

A limit concerns nearby values, not necessarily the value at the point.

# Factorization reveals the nearby behaviour

---

Whenever  $x \neq 1$ ,

$$\frac{x^2 - 1}{x - 1}$$

# Factorization reveals the nearby behaviour

---

Whenever  $x \neq 1$ ,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

# Factorization reveals the nearby behaviour

---

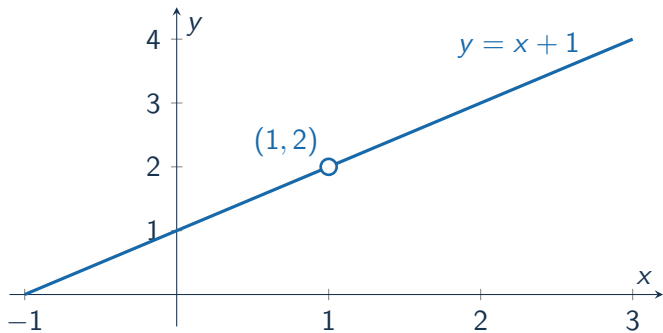
Whenever  $x \neq 1$ ,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1.$$

Therefore, as  $x$  approaches 1, the value of the function approaches

$$1 + 1 = 2.$$

# The missing value does not prevent the limit



$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

# Function value and limit answer different questions

---

## Function value

$f(a)$  describes what happens **at** the point.

## Limit

$\lim_{x \rightarrow a} f(x)$  describes what happens **around** the point.

A limit may exist even when  $f(a)$  is undefined or has a different value.

# Sequences provide another way to test a limit

---

Suppose  $x_n \neq a$  and  $x_n \rightarrow a$ .

If

$$\lim_{x \rightarrow a} f(x) = L,$$

then every such sequence must satisfy

$$f(x_n) \rightarrow L.$$

Consequently, two approaches producing different limiting values prove that the limit does not exist.

## Two sequences expose the oscillation of $\sin(1/x)$

---

Let

$$x_n = \frac{1}{\pi/2 + 2\pi n}, \quad y_n = \frac{1}{3\pi/2 + 2\pi n}.$$

Both sequences tend to 0, but

$$\sin(1/x_n) = 1, \quad \sin(1/y_n) = -1.$$

$\lim_{x \rightarrow 0} \sin(1/x)$  does not exist.

# Changing one value cannot change a limit

---

Let

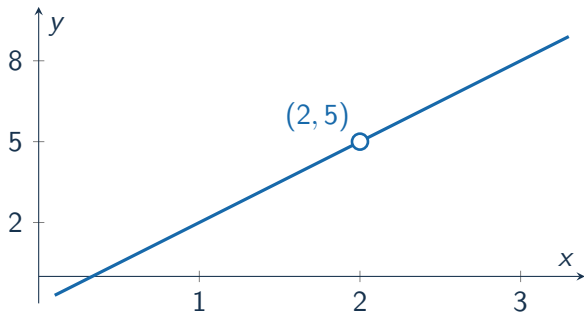
$$f(x) = \begin{cases} 3x - 1, & x \neq 2, \\ 100, & x = 2. \end{cases}$$

Near  $x = 2$ , but away from 2, the function agrees exactly with  $3x - 1$ .

$$3x - 1 \longrightarrow 5 \quad \text{as} \quad x \longrightarrow 2.$$

# The isolated value is irrelevant

---



At the point

$$f(2) = 100.$$

Around the point

$$f(x) = 3x - 1.$$

## One exceptional value cannot affect nearby behaviour

---

$$\lim_{x \rightarrow 2} f(x) = 5, \quad f(2) = 100.$$

The limit is 5, even though the function value is 100.

# The epsilon-delta definition

# The definition turns closeness into inequalities

---

We write

$$\lim_{x \rightarrow a} f(x) = L$$

if

# The definition turns closeness into inequalities

---

We write

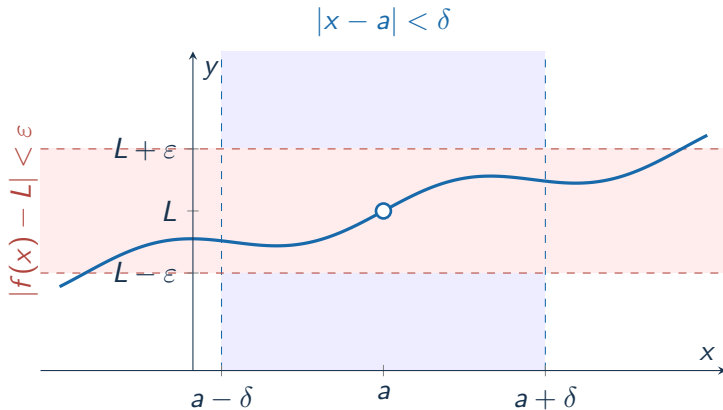
$$\lim_{x \rightarrow a} f(x) = L$$

if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$x \in A, \quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

$\varepsilon$  controls the output;  $\delta$  controls the input.

# A delta-window must enter the epsilon-band



## Every part of the definition has a role

---

- ★ *For every  $\varepsilon > 0$ :* the requested accuracy may be arbitrarily small.

# Every part of the definition has a role

---

- ★ *For every  $\varepsilon > 0$ :* the requested accuracy may be arbitrarily small.
- ★ *There exists  $\delta > 0$ :* the input window may depend on  $\varepsilon$ .

# Every part of the definition has a role

---

- ★ *For every  $\varepsilon > 0$ :* the requested accuracy may be arbitrarily small.
- ★ *There exists  $\delta > 0$ :* the input window may depend on  $\varepsilon$ .
- ★  $0 < |x - a|$ : the value at  $a$  is excluded.

# Every part of the definition has a role

---

- ★ *For every  $\varepsilon > 0$ :* the requested accuracy may be arbitrarily small.
- ★ *There exists  $\delta > 0$ :* the input window may depend on  $\varepsilon$ .
- ★  $0 < |x - a|$ : the value at  $a$  is excluded.
- ★  $|f(x) - L| < \varepsilon$ : all nearby outputs lie in the epsilon-band.

## Example: prove a linear limit directly

---

We want to prove

$$\lim_{x \rightarrow 3} (2x - 1) = 5.$$

Given  $\varepsilon > 0$ , our target is

$$|(2x - 1) - 5| < \varepsilon.$$

The question is: how small should  $|x - 3|$  be?

## Rewrite the output error using the input error

---

$$\begin{aligned}|(2x - 1) - 5| &= |2x - 6| \\ &= 2|x - 3|.\end{aligned}$$

Therefore,

$$2|x - 3| < \varepsilon$$

is guaranteed whenever

$$|x - 3| < \frac{\varepsilon}{2}.$$

# The calculation tells us how to choose delta

---

Choose

$$\delta = \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} 0 < |x - 3| < \delta &\implies |(2x - 1) - 5| \\ &= 2|x - 3| \\ &< 2\delta = \varepsilon. \end{aligned}$$

$$\lim_{x \rightarrow 3} (2x - 1) = 5.$$

# An epsilon-delta proof follows a reusable pattern

---

1. Start with the desired inequality  $|f(x) - L| < \varepsilon$ .

# An epsilon-delta proof follows a reusable pattern

---

1. Start with the desired inequality  $|f(x) - L| < \varepsilon$ .
2. Rewrite or estimate it using  $|x - a|$ .

# An epsilon-delta proof follows a reusable pattern

---

1. Start with the desired inequality  $|f(x) - L| < \varepsilon$ .
2. Rewrite or estimate it using  $|x - a|$ .
3. Choose  $\delta$  from that estimate.

# An epsilon-delta proof follows a reusable pattern

---

1. Start with the desired inequality  $|f(x) - L| < \varepsilon$ .
2. Rewrite or estimate it using  $|x - a|$ .
3. Choose  $\delta$  from that estimate.
4. Verify the implication from the definition.

Work backwards to find  $\delta$ , then write the proof forwards.

# Limit laws and squeezing

# Finite limits obey the usual arithmetic rules

---

If

$$f(x) \longrightarrow L, \quad g(x) \longrightarrow M,$$

then

$$f(x) + g(x) \longrightarrow L + M,$$

$$f(x)g(x) \longrightarrow LM,$$

$$\frac{f(x)}{g(x)} \longrightarrow \frac{L}{M} \quad (M \neq 0).$$

Constants, integer powers and admissible roots also pass through limits.

# Polynomials and rational functions allow substitution

---

For every polynomial  $P$ ,

$$\lim_{x \rightarrow a} P(x) = P(a).$$

For a rational function,

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

provided that

$$Q(a) \neq 0.$$

## Direct substitution works when the denominator survives

---

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^3 + 4x - 1}{x^2 + 1} &= \frac{2^3 + 4 \cdot 2 - 1}{2^2 + 1} \\ &= \frac{15}{5} \\ &= 3.\end{aligned}$$

There is no indeterminate form because the denominator tends to 5, not to 0.

# The squeeze theorem traps a difficult function

---

Suppose that, near  $a$ ,

$$g(x) \leq f(x) \leq h(x),$$

and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L.$$

Then

$$\lim_{x \rightarrow a} f(x) = L.$$

# Bounded oscillation can still have a limit

---

Since

$$|\sin(1/x)| \leq 1,$$

we have

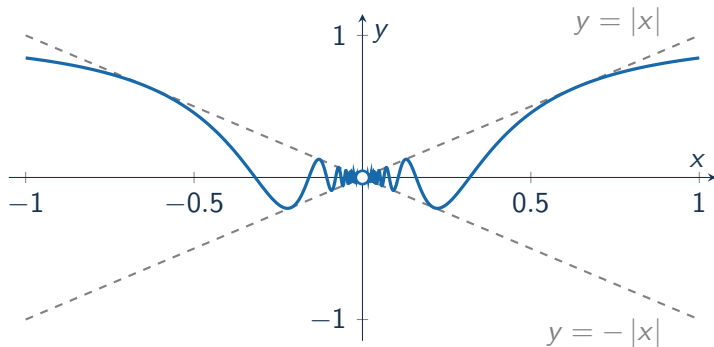
$$|x \sin(1/x)| \leq |x|.$$

Equivalently,

$$-|x| \leq x \sin(1/x) \leq |x|.$$

Both outer functions tend to 0 as  $x \rightarrow 0$ .

# The oscillations remain inside a shrinking envelope



$$\lim_{x \rightarrow 0} x \sin(1/x) = 0.$$

# One-sided limits

# One-sided limits remember the direction of approach

---

The left-hand limit

$$\lim_{x \rightarrow a^-} f(x)$$

uses only points  $x < a$ .

The right-hand limit

$$\lim_{x \rightarrow a^+} f(x)$$

uses only points  $x > a$ .

# A two-sided limit requires agreement from both sides

---

If the domain approaches  $a$  from both sides, then

$$\lim_{x \rightarrow a} f(x) = L$$

if and only if

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L.$$

The left and right limits must both exist and be equal.

# At an endpoint only one direction is available

---

For a function defined on  $[a, b]$ :

- At  $a$ , we approach from the right:

$$\lim_{x \rightarrow a^+} f(x).$$

- At  $b$ , we approach from the left:

$$\lim_{x \rightarrow b^-} f(x).$$

## Example: inspect each branch separately

---

Consider

$$f(x) = \begin{cases} x^2, & x < 1, \\ 2x + 1, & x \geq 1. \end{cases}$$

To find the limit at 1, use:

- $x^2$  on the left of 1;
- $2x + 1$  on the right of 1.

## The two branches approach different values

---

From the left,

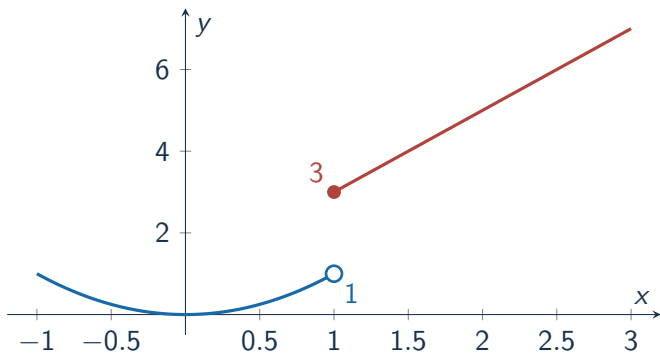
$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1.$$

From the right,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x + 1) = 3.$$

The two values do not agree.

The graph displays a jump at  $x = 1$



$\lim_{x \rightarrow 1} f(x)$  does not exist.

# The value at the jump cannot repair the limit

---

For the piecewise function,

$$f(1) = 3.$$

However,

$$\lim_{x \rightarrow 1^-} f(x) = 1 \neq 3 = \lim_{x \rightarrow 1^+} f(x).$$

A two-sided limit fails whenever the one-sided limits disagree.

# Infinite limits and limits at infinity

# An infinite limit describes unbounded behaviour

---

The notation

$$\lim_{x \rightarrow a} f(x) = +\infty$$

means that  $f(x)$  becomes larger than every prescribed positive number when  $x$  is sufficiently close to  $a$ .

$+\infty$  is not a real number; it describes a type of divergence.

# A reciprocal changes sign across its vertical asymptote

---

For

$$f(x) = \frac{1}{x-2},$$

we have

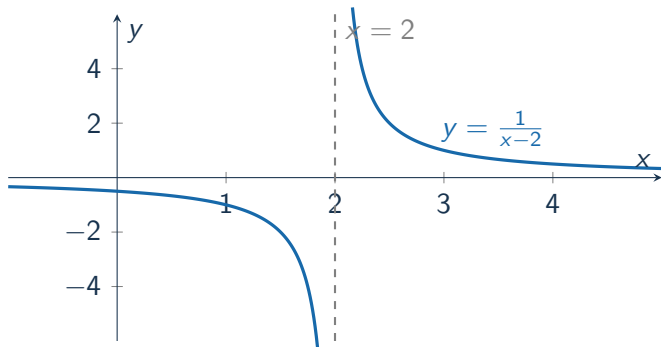
$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$$

and

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = +\infty.$$

# The graph separates the two infinite behaviours

---



## Opposite signs prevent a two-sided infinite limit

---

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty, \quad \lim_{x \rightarrow 2^+} \frac{1}{x-2} = +\infty.$$

The two-sided limit does not exist.

Even for infinite limits, the two sides must describe the same behaviour.

# A limit at infinity describes long-term behaviour

---

The notation

$$x \rightarrow +\infty$$

means that  $x$  increases without bound.

For a rational expression, divide the numerator and denominator by the highest relevant power of  $x$ .

This reveals which terms survive and which terms tend to zero.

# The main families have a hierarchy of growth

---

For every  $p > 0$  and  $a > 1$ ,

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x^p} = 0,$$

so powers grow faster than logarithms, and

$$\lim_{x \rightarrow +\infty} \frac{x^p}{a^x} = 0,$$

so exponentials grow faster than powers.

logarithm   <   power   <   exponential.

## Example: divide by the dominant power

---

Consider

$$\lim_{x \rightarrow +\infty} \frac{3x^2 - \sin x}{2x^2 + 1}.$$

Dividing numerator and denominator by  $x^2$  gives

$$\frac{3 - \sin(x)/x^2}{2 + 1/x^2}.$$

It remains to understand the two small terms.

# Boundedness removes the oscillating term

---

Since  $|\sin x| \leq 1$ ,

$$0 \leq \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}.$$

As  $x \rightarrow +\infty$ ,

$$\frac{1}{x^2} \longrightarrow 0.$$

Therefore, by squeezing,

$$\frac{\sin x}{x^2} \longrightarrow 0.$$

# Only the leading coefficients remain

---

Now

$$\frac{3 - \sin(x)/x^2}{2 + 1/x^2} \longrightarrow \frac{3 - 0}{2 + 0}.$$

$$\lim_{x \rightarrow +\infty} \frac{3x^2 - \sin x}{2x^2 + 1} = \frac{3}{2}.$$

Indeterminate forms

# An indeterminate form is a signal, not an answer

---

Substitution may produce one of the symbolic forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0 \cdot \infty,$$

or, for powers,

$$1^{\infty}, \quad 0^0, \quad \infty^0.$$

These forms do not determine the limit. They tell us that the expression must be transformed before a limit law can be applied.

# One 0/0 form, several possible limits

---

As  $x \rightarrow 0$ ,

$$\frac{x}{x} \rightarrow 1, \quad \frac{x^2}{x} \rightarrow 0, \quad \frac{|x|}{x} \text{ has no two-sided limit.}$$

All three expressions initially have the same form 0/0, but their limits are different.

An indeterminate form describes missing information, not a value.

# Choose a transformation that matches the expression

---

## Algebraic expressions

- factor and cancel;
- take a common denominator;
- divide by the dominant power;
- multiply by a conjugate.

## Products and powers

- rewrite a product as a quotient;
- use logarithms for variable powers;
- use a standard limit;
- squeeze when a factor is bounded.

## Example: subtraction creates an $\infty - \infty$ form

---

Evaluate

$$\lim_{x \rightarrow +\infty} \left( \sqrt{x^2 + 3x} - x \right).$$

Direct inspection gives the form

$$\infty - \infty.$$

The square root suggests multiplying by the conjugate.

## Multiply by the conjugate

---

$$\begin{aligned}\sqrt{x^2 + 3x} - x &= \left(\sqrt{x^2 + 3x} - x\right) \frac{\sqrt{x^2 + 3x} + x}{\sqrt{x^2 + 3x} + x} \\ &= \frac{x^2 + 3x - x^2}{\sqrt{x^2 + 3x} + x} \\ &= \frac{3x}{\sqrt{x^2 + 3x} + x}.\end{aligned}$$

The subtraction has disappeared.

## Divide numerator and denominator by $x$

---

For  $x > 0$ ,

$$\frac{3x}{\sqrt{x^2 + 3x} + x} = \frac{3}{\sqrt{1 + 3/x} + 1}.$$

Now ordinary substitution is valid:

$$\frac{3}{\sqrt{1 + 3/x} + 1} \longrightarrow \frac{3}{1 + 1}.$$

$$\lim_{x \rightarrow +\infty} (\sqrt{x^2 + 3x} - x) = \frac{3}{2}.$$

Standard limits

# A small collection of limits solves many problems

---

As  $x \rightarrow 0$ ,

$$\frac{\sin x}{x} \rightarrow 1, \quad \frac{1 - \cos x}{x^2} \rightarrow \frac{1}{2}, \quad \frac{\tan x}{x} \rightarrow 1.$$

These trigonometric formulas require angles to be measured in **radians**.

# Exponential and logarithmic limits

---

As  $x \rightarrow 0$ ,

$$\frac{e^x - 1}{x} \rightarrow 1, \quad \frac{a^x - 1}{x} \rightarrow \ln a \quad (a > 0),$$

and

$$\frac{\ln(1+x)}{x} \rightarrow 1.$$

For variable powers,

$$(1+x)^{1/x} \rightarrow e.$$

Example: identify two copies of  $\sin u/u$

---

Evaluate

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)}.$$

Both numerator and denominator tend to zero. Insert the useful factors  $2x$  and  $3x$ :

$$\frac{e^{2x} - 1}{\sin(3x)} = \frac{e^{2x} - 1}{2x} \frac{3x}{\sin(3x)} \frac{2}{3}.$$

## Apply the standard limits factor by factor

---

When  $x \rightarrow 0$ , both  $2x \rightarrow 0$  and  $3x \rightarrow 0$ . Hence

$$\frac{e^{2x} - 1}{2x} \rightarrow 1, \quad \frac{3x}{\sin(3x)} \rightarrow 1.$$

Therefore,

$$\frac{e^{2x} - 1}{\sin(3x)} \rightarrow 1 \cdot 1 \cdot \frac{2}{3}.$$

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)} = \frac{2}{3}.$$

## Example: shift the logarithmic standard limit

---

Evaluate

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}.$$

Set

$$u = x - 1.$$

Then  $u \rightarrow 0$  and  $x = 1 + u$ , so

$$\frac{\ln x}{x - 1} = \frac{\ln(1 + u)}{u}.$$

# The substitution reveals the known pattern

---

The standard limit gives

$$\frac{\ln(1+u)}{u} \longrightarrow 1 \quad (u \rightarrow 0).$$

Therefore,

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = 1.$$

The variable name does not matter; the local pattern does.

## Example: a variable power

---

Evaluate

$$\lim_{x \rightarrow 0} (1 + 3x)^{2/x}.$$

The base tends to 1 while the exponent becomes unbounded, so this has the indeterminate form  $1^\infty$ .

The model limit is

$$(1 + u)^{1/u} \longrightarrow e.$$

## Rewrite the expression around the model limit

---

Set  $u = 3x$ . Then  $u \rightarrow 0$  and

$$\frac{2}{x} = \frac{6}{u}.$$

Thus

$$\begin{aligned}(1 + 3x)^{2/x} &= (1 + u)^{6/u} \\ &= \left( (1 + u)^{1/u} \right)^6.\end{aligned}$$

Taking the limit gives

$$\left( (1 + u)^{1/u} \right)^6 \longrightarrow e^6.$$

$$\lim_{x \rightarrow 0} (1 + 3x)^{2/x} = e^6.$$

# Closing summary

# A practical strategy for limits

---

1. Try direct substitution.

# A practical strategy for limits

---

1. Try direct substitution.
2. If it works, use the limit laws.

# A practical strategy for limits

---

1. Try direct substitution.
2. If it works, use the limit laws.
3. If it gives an indeterminate form, transform the expression.

# A practical strategy for limits

---

1. Try direct substitution.
2. If it works, use the limit laws.
3. If it gives an indeterminate form, transform the expression.
4. Compare with a standard limit or use the squeeze theorem.

# A practical strategy for limits

---

1. Try direct substitution.
2. If it works, use the limit laws.
3. If it gives an indeterminate form, transform the expression.
4. Compare with a standard limit or use the squeeze theorem.
5. Check the two sides separately near jumps and asymptotes.

# Final check

---

Suppose

$$\lim_{x \rightarrow a^-} f(x) = 2, \quad \lim_{x \rightarrow a^+} f(x) = 2, \quad f(a) = 7.$$

What is  $\lim_{x \rightarrow a} f(x)$ ?

# Final check

---

Suppose

$$\lim_{x \rightarrow a^-} f(x) = 2, \quad \lim_{x \rightarrow a^+} f(x) = 2, \quad f(a) = 7.$$

What is  $\lim_{x \rightarrow a} f(x)$ ?

The one-sided limits agree, so

$$\boxed{\lim_{x \rightarrow a} f(x) = 2.}$$

The isolated value  $f(a) = 7$  does not change the limit.