Calculus 1
Midterm 1

Version A

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2 + 2x + 1, & x \leqslant 0, \\ ax + b, & 0 < x < 2, \\ (x-2)^2 + 5, & x \geqslant 2. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, show whether f is differentiable at $x = 0$ and at $x = 2$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. Consider the equation $\cos x = x$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(0, \pi/2)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = \dfrac{x}{1+x^2}$ on the closed interval $[-2, 2]$.

(a) **[0.7 pts]** Find all critical points of f in $(-2, 2)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[-2, 2]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[-2, 2]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If a function f has a local maximum at $x = c$ and $f''(c)$ exists, then $f''(c) < 0$.

Midterm 1 – Solutions

Version A

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 2. At $x = 0$,

$$f(0) = 1, \qquad \lim_{x \to 0^+} f(x) = b.$$

Thus, continuity at 0 requires $b = 1$. At $x = 2$,

$$\lim_{x \to 2^-} f(x) = 2a + b, \qquad f(2) = 5.$$

Therefore,

$$2a + b = 5.$$

Using $b = 1$, we obtain

$$\boxed{a = 2, \qquad b = 1.}$$

(b) For these values,

$$f(x) = \begin{cases} x^2 + 2x + 1, & x \leqslant 0, \\ 2x + 1, & 0 < x < 2, \\ (x-2)^2 + 5, & x \geqslant 2. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = (2x+2)|_{x=0} = 2, \qquad f'_+(0) = 2.$$

They agree, so f is differentiable at 0 and $f'(0) = 2$.

At $x = 2$,

$$f'_-(2) = 2, \qquad f'_+(2) = 2(x-2)|_{x=2} = 0.$$

The one-sided derivatives are different, so f is not differentiable at 2.

(c) Since $f(0) = 1$ and $f'(0) = 2$, the tangent line is

$$y - f(0) = f'(0)(x - 0),$$

and hence

$$\boxed{y = 2x + 1.}$$

Problem 2.

(a) The function

$$g(x) = \cos x - x$$

is continuous on $\mathbb{R}$ and, in particular, on $[0, \pi/2]$. We have

$$g(0) = 1 > 0, \qquad g(\pi/2) = -\frac{\pi}{2} < 0.$$

Because g is continuous on $[0, \pi/2]$ and 0 lies between $g(0)$ and $g(\pi/2)$, the Intermediate Value Theorem gives a number $c \in (0, \pi/2)$ such that

$$g(c) = 0.$$

Thus, the equation has at least one solution in $(0, \pi/2)$.

(b) At the endpoints of the smaller interval $(0, 1)$, we obtain

$$g(0) = 1 > 0,$$
$$g(1) = \cos 1 - 1 < 0.$$

Here $\cos 1 < 1$ because $1 \neq 0$ and cosine is strictly smaller than 1 on $(0, \pi/2]$. Since g is continuous on $[0, 1]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(0, 1).}$$

(c) Any real solution of $\cos x = x$ must satisfy $-1 \leqslant x \leqslant 1$. On this interval cosine is positive, so a solution must in fact belong to $(0, 1)$. Moreover,

$$g'(x) = -\sin x - 1 < 0$$

for every $x \in (0, 1)$. Thus, g is strictly decreasing on $(0, 1)$ and can vanish there at most once. Combined with part **(c)**, this proves that the solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2}.$$

Since the denominator is always positive, the critical numbers in $(-2, 2)$ are precisely the solutions of $f'(x) = 0$:

$$\boxed{x = -1 \quad \text{and} \quad x = 1.}$$

(b) The sign of f' is determined by $1 - x^2$:

x	$(-2, -1)$	-1	$(-1, 1)$	1	$(1, 2)$
$f'(x)$	$-$	0	$+$	0	$-$
f	decreasing		increasing		decreasing

Hence, f is decreasing on $[-2, -1]$ and $[1, 2]$, and increasing on $[-1, 1]$. Moreover,

$$f(-1) = -\frac{1}{2}, \qquad f(1) = \frac{1}{2}.$$

Therefore, $(-1, -1/2)$ is a local minimum point and $(1, 1/2)$ is a local maximum point.

(c) On a closed interval, absolute extrema may occur at critical numbers or endpoints. We compare

x	-2	-1	1	2
$f(x)$	$-2/5$	$-1/2$	$1/2$	$2/5$

Thus,

$$\boxed{\min_{[-2,2]} f = -\frac{1}{2} \text{ at } x = -1,} \qquad \boxed{\max_{[-2,2]} f = \frac{1}{2} \text{ at } x = 1.}$$

(d) The second derivative is

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

The denominator is positive, so the sign is determined by $x(x^2-3)$. Consequently, f is concave down on

$$(-2, -\sqrt{3}) \cup (0, \sqrt{3})$$

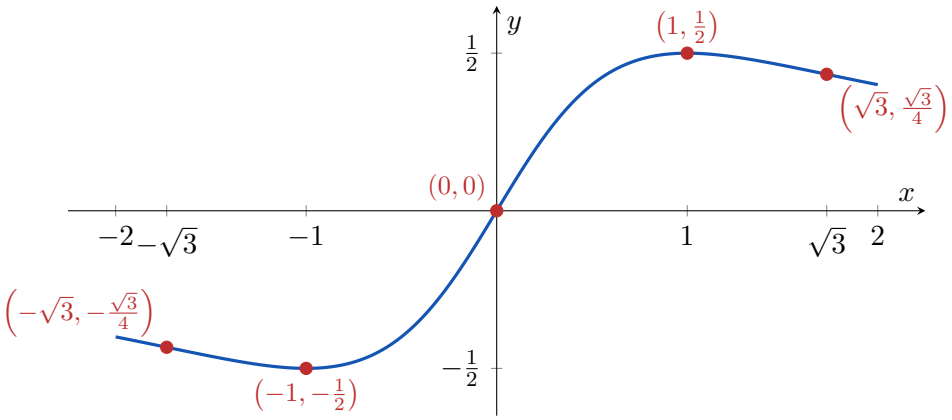
and concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, 2).$$

The concavity changes at $-\sqrt{3}$, 0 and $\sqrt{3}$. Therefore, the inflection points are

$$\boxed{\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).}$$

(e) The numerator vanishes only at $x = 0$, so the graph meets both coordinate axes at $(0,0)$. Together with the information obtained above, this gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = -\sin^4 x.$$

Since $\sin^4 x \geqslant 0$ for every x, we have

$$f(x) \leqslant 0 = f(0).$$

Thus, f has a local maximum at $x = 0$. Differentiating twice gives

$$f'(x) = -4\sin^3 x \cos x$$

and

$$f''(x) = -12\sin^2 x \cos^2 x + 4\sin^4 x.$$

Therefore,

$$f''(0) = 0,$$

not $f''(0) < 0$. This counterexample disproves the statement.

Calculus 1
Midterm 1

Version B

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2, & x \leqslant -1, \\ ax + b, & -1 < x < 1, \\ (x-1)^2 - 3, & x \geqslant 1. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = -1$ and at $x = 1$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = -1$.

Problem 2. Consider the equation $e^x = 3 - x$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(0, 1)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = \dfrac{x^2}{1 + x^2}$ on the closed interval $[-2, 2]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(-2, 2)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[-2, 2]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[-2, 2]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If $f''(x) > 0$ at every point of an interval, then f is increasing on that interval.

Midterm 1 – Solutions

Version B

Problem 1.

(a) Continuity only needs to be checked at the junction points -1 and 1. At $x = -1$,

$$f(-1) = 1, \qquad \lim_{x \to -1^+} f(x) = -a + b.$$

Thus, continuity at -1 requires

$$-a + b = 1.$$

At $x = 1$,

$$\lim_{x \to 1^-} f(x) = a + b, \qquad f(1) = -3.$$

Therefore,

$$a + b = -3.$$

Solving the system gives

$$\boxed{a = -2, \qquad b = -1.}$$

(b) For these values,

$$f(x) = \begin{cases} x^2, & x \leqslant -1, \\ -2x - 1, & -1 < x < 1, \\ (x-1)^2 - 3, & x \geqslant 1. \end{cases}$$

At $x = -1$, the one-sided derivatives are

$$f'_-(-1) = 2x|_{x=-1} = -2, \qquad f'_+(-1) = -2.$$

They agree, so f is differentiable at -1 and $f'(-1) = -2$.

At $x = 1$,

$$f'_-(1) = -2, \qquad f'_+(1) = 2(x-1)|_{x=1} = 0.$$

The one-sided derivatives are different, so f is not differentiable at 1.

(c) Since $f(-1) = 1$ and $f'(-1) = -2$, the tangent line is

$$y - 1 = -2(x + 1),$$

or, equivalently,

$$\boxed{y = -2x - 1.}$$

Problem 2.

(a) Define

$$g(x) = e^x + x - 3.$$

The exponential function and every polynomial are continuous on $\mathbb{R}$. Therefore, g is continuous on $\mathbb{R}$ and, in particular, on the closed interval $[0, 1]$. At the endpoints,

$$g(0) = -2 < 0, \qquad g(1) = e - 2 > 0.$$

Because g is continuous on $[0, 1]$ and changes sign there, the Intermediate Value Theorem gives a number $c \in (0, 1)$ such that

$$g(c) = 0.$$

Thus, $e^c = 3 - c$, so the original equation has at least one solution in $(0, 1)$.

(b) At the endpoints of the smaller interval $(\ln(2), 1)$, we have

$$g(\ln 2) = e^{\ln 2} + \ln 2 - 3 = \ln 2 - 1 < 0,$$
$$g(1) = e - 2 > 0.$$

Here $\ln 2 < 1$ because $2 < e$. Since g is continuous on $[\ln 2, 1]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(\ln 2, 1).}$$

(c) We have

$$g'(x) = e^x + 1 > 0$$

for every $x \in \mathbb{R}$. Hence, g is strictly increasing on $\mathbb{R}$ and can vanish at most once. Together with the existence proved above, this shows that the solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{2x(1 + x^2) - 2x^3}{(1 + x^2)^2} = \frac{2x}{(1 + x^2)^2}.$$

The denominator is always positive, so the only critical number in $(-2, 2)$ is

$$\boxed{x = 0.}$$

(b) The sign of f' is the sign of x:

x	$(-2, 0)$	0	$(0, 2)$
$f'(x)$	$-$	0	$+$
f	decreasing		increasing

Hence, f is decreasing on $[-2, 0]$ and increasing on $[0, 2]$. Therefore,

$$\boxed{(0, 0) \text{ is a local minimum point.}}$$

There is no local maximum in the open interval $(-2, 2)$.

(c) For the absolute extrema, we compare the critical number and the endpoints:

x	-2	0	2
$f(x)$	$4/5$	0	$4/5$

Thus,

$$\boxed{\min_{[-2,2]} f = 0 \text{ at } x = 0,}$$

and

$$\boxed{\max_{[-2,2]} f = \frac{4}{5} \text{ at } x = -2 \text{ and } x = 2.}$$

(d) Differentiating again gives

$$f''(x) = \frac{2(1-3x^2)}{(1+x^2)^3}.$$

The denominator is positive. The numerator vanishes when

$$x = \pm\frac{1}{\sqrt{3}}.$$

Consequently, f is concave down on

$$\left(-2, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 2\right)$$

and concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

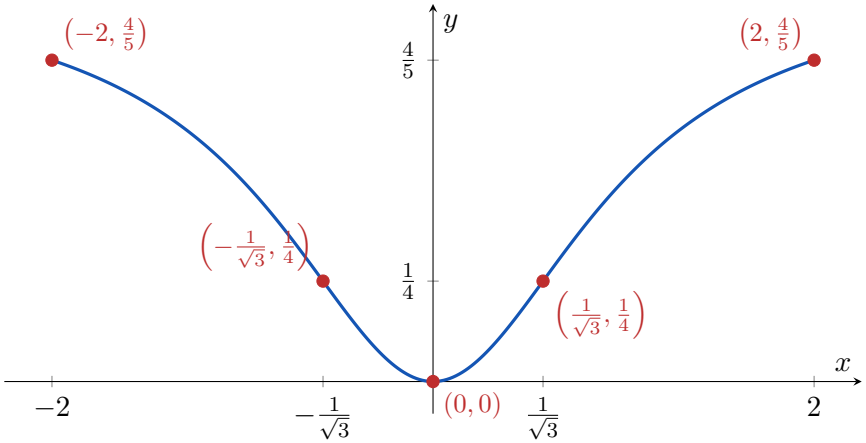
Since the concavity changes at both values and

$$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{1}{4},$$

the inflection points are

$$\boxed{\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right).}$$

(e) The numerator vanishes only at $x = 0$, so the graph meets both coordinate axes only at $(0, 0)$. The function is even, so its graph is symmetric about the y-axis. Together with the preceding information, this gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = e^{-x}.$$

Then

$$f'(x) = -e^{-x} < 0, \qquad f''(x) = e^{-x} > 0$$

for every $x \in \mathbb{R}$. Thus, f is strictly decreasing on $\mathbb{R}$ even though its second derivative is positive everywhere. Positive second derivative means that the function is concave up; it does not mean that the function is increasing.

Calculus 1
Midterm 1

Version C

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2 - 2x, & x \leqslant 0, \\ ax + b, & 0 < x < 3, \\ (x-3)^2 - 6, & x \geqslant 3. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 3$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. Consider the equation $e^{-x} = x$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(0, 1)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = 4 - (\ln x - 1)^2$ on the closed interval $[1, e^3]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(1, e^3)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[1, e^3]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[1, e^3]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If a differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(x) = 0$ for infinitely many values of x, then f is periodic.

Midterm 1 – Solutions

Version C

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 3. At $x = 0$,

$$f(0) = 0, \qquad \lim_{x\to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 0.$$

At $x = 3$,

$$\lim_{x\to 3^-} f(x) = 3a + b, \qquad f(3) = -6.$$

Therefore,

$$3a + b = -6.$$

Using $b = 0$, we obtain

$$\boxed{a = -2, \qquad b = 0.}$$

(b) For these values,

$$f(x) = \begin{cases} x^2 - 2x, & x \leqslant 0, \\ -2x, & 0 < x < 3, \\ (x-3)^2 - 6, & x \geqslant 3. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = (2x-2)|_{x=0} = -2, \qquad f'_+(0) = -2.$$

They agree, so f is differentiable at 0 and $f'(0) = -2$.

At $x = 3$,

$$f'_-(3) = -2, \qquad f'_+(3) = 2(x-3)|_{x=3} = 0.$$

The one-sided derivatives are different, so f is not differentiable at 3.

(c) Since $f(0) = 0$ and $f'(0) = -2$, the tangent line is

$$y - 0 = -2(x - 0),$$

and hence

$$\boxed{y = -2x.}$$

Problem 2.

(a) Define

$$g(x) = e^{-x} - x.$$

The function g is continuous on $\mathbb{R}$ and, in particular, on $[0,1]$. Moreover,

$$g(0) = 1 > 0, \qquad g(1) = e^{-1} - 1 < 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (0,1)$ such that $g(c) = 0$. Hence, $e^{-c} = c$, so the original equation has at least one solution in $(0,1)$.

(b) At the endpoints of the smaller interval,

$$g\left(\frac{1}{2}\right) = e^{-1/2} - \frac{1}{2} > 0, \qquad g(1) = e^{-1} - 1 < 0.$$

Indeed, $e < 4$, and hence $e^{-1/2} > 1/2$. Since g is continuous on $[1/2, 1]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(1/2, 1).}$$

(c) We have

$$g'(x) = -e^{-x} - 1 < 0$$

for every $x \in \mathbb{R}$. Hence, g is strictly decreasing on $\mathbb{R}$ and can vanish at most once. Together with part **(a)**, this proves that the solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{2(1 - \ln x)}{x}.$$

Since $x > 0$, the derivative vanishes precisely when $\ln x = 1$. Hence, the only critical number in $(1, e^3)$ is

$$\boxed{x = e.}$$

(b) The sign chart for f' is

x	$(1, e)$	e	(e, e^3)
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{(e, 4) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	1	e	e^3
$f(x)$	3	4	0

Thus,

$$\boxed{\max_{[1,e^3]} f = 4 \text{ at } x = e,} \qquad \boxed{\min_{[1,e^3]} f = 0 \text{ at } x = e^3.}$$

(d) Differentiating again gives

$$f''(x) = \frac{2(\ln x - 2)}{x^2}.$$

The second derivative vanishes precisely when

$$\ln x = 2, \qquad \text{that is, when} \qquad x = e^2.$$

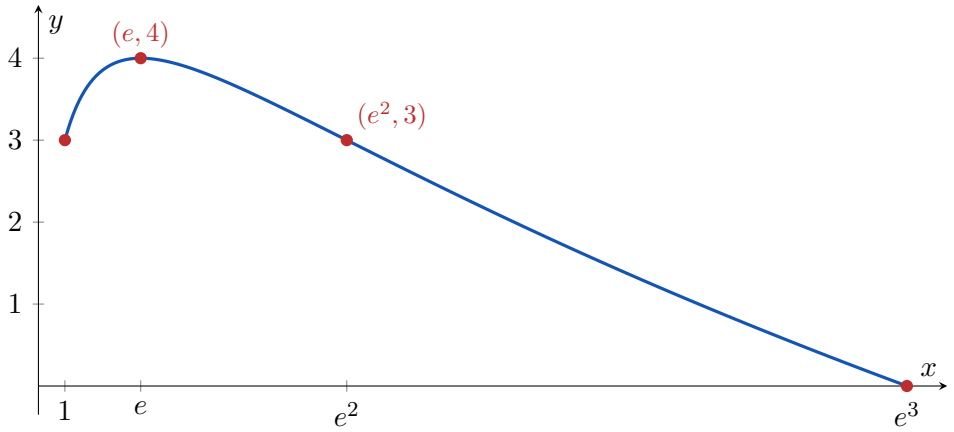
Consequently, f is concave down on $(1, e^2)$ and concave up on (e^2, e^3). Since the concavity changes at e^2 and

$$f(e^2) = 4 - (2-1)^2 = 3,$$

the unique inflection point is

$$\boxed{(e^2, 3).}$$

(e) On the interval $[1, e^3]$, the graph meets the x-axis only at $(e^3, 0)$. Together with the preceding information, this gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = x + \sin x.$$

This function is differentiable and

$$f'(x) = 1 + \cos x.$$

Therefore,

$$f'((2k+1)\pi) = 0 \qquad \text{for every } k \in \mathbb{Z},$$

so the derivative vanishes at infinitely many points. However, f is not periodic. Indeed, it is unbounded, while every continuous periodic function on $\mathbb{R}$ is bounded. Thus, having infinitely many zeros of the derivative does not imply periodicity.

Calculus 1
Midterm 1

Version D

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} x^2, & x \leqslant 1, \\ ax + b, & 1 < x < 3, \\ (x-3)^2 + 5, & x \geqslant 3. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 1$ and at $x = 3$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 1$.

Problem 2. Consider the equation $e^x + x = 3$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(0, 1)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = \sin^2 x$ on the closed interval $[0, \pi]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(0, \pi)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[0, \pi]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[0, \pi]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If f is twice differentiable and $f''(c) = 0$, then $(c, f(c))$ is an inflection point of the graph of f.

Midterm 1 – Solutions

Version D

Problem 1.

(a) Continuity only needs to be checked at the junction points 1 and 3. At $x = 1$,

$$f(1) = 1, \qquad \lim_{x\to 1^+} f(x) = a + b.$$

Thus, continuity at 1 requires

$$a + b = 1.$$

At $x = 3$,

$$\lim_{x\to 3^-} f(x) = 3a + b, \qquad f(3) = 5.$$

Therefore,

$$3a + b = 5.$$

Subtracting the two equations gives $2a = 4$. Hence,

$$\boxed{a = 2, \qquad b = -1.}$$

(b) For these values,

$$f(x) = \begin{cases} x^2, & x \leqslant 1, \\ 2x - 1, & 1 < x < 3, \\ (x-3)^2 + 5, & x \geqslant 3. \end{cases}$$

At $x = 1$, the one-sided derivatives are

$$f'_-(1) = 2x|_{x=1} = 2, \qquad f'_+(1) = 2.$$

They agree, so f is differentiable at 1 and $f'(1) = 2$.

At $x = 3$,

$$f'_-(3) = 2, \qquad f'_+(3) = 2(x-3)|_{x=3} = 0.$$

The one-sided derivatives are different, so f is not differentiable at 3.

(c) Since $f(1) = 1$ and $f'(1) = 2$, the tangent line is

$$y - 1 = 2(x - 1),$$

and hence

$$\boxed{y = 2x - 1.}$$

Problem 2.

(a) Define

$$g(x) = e^x + x - 3.$$

The function g is continuous on $\mathbb{R}$ and, in particular, on $[0,1]$. Moreover,

$$g(0) = -2 < 0, \qquad g(1) = e - 2 > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (0,1)$ such that $g(c) = 0$. Hence, $e^c + c = 3$, so the original equation has at least one solution in $(0,1)$.

(b) At the endpoints of the smaller interval $(1/2, 1)$, we have

$$g\left(\frac{1}{2}\right) = \sqrt{e} + \frac{1}{2} - 3 < 0, \qquad g(1) = e - 2 > 0.$$

Indeed, $e < 4$, and hence $\sqrt{e} < 2$. Since g is continuous on $[1/2, 1]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(1/2, 1).}$$

(c) We have

$$g'(x) = e^x + 1 > 0$$

for every $x \in \mathbb{R}$. Hence, g is strictly increasing on $\mathbb{R}$ and can vanish at most once. Together with part **(a)**, this proves that the solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = 2 \sin x \cos x = \sin(2x).$$

The only zero of $\sin(2x)$ in $(0, \pi)$ is obtained when $2x = \pi$. Hence, the only critical number is

$$\boxed{x = \frac{\pi}{2}.}$$

(b) The sign chart for f' is

x	$(0, \pi/2)$	$\pi/2$	$(\pi/2, \pi)$
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{\left(\frac{\pi}{2}, 1\right) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	0	$\pi/2$	π
$f(x)$	0	1	0

Thus,

$$\boxed{\max_{[0,\pi]} f = 1 \text{ at } x = \frac{\pi}{2},} \qquad \boxed{\min_{[0,\pi]} f = 0 \text{ at } x = 0 \text{ and } x = \pi.}$$

(d) Differentiating again gives

$$f''(x) = 2\cos(2x).$$

The second derivative vanishes in $(0, \pi)$ when

$$x = \frac{\pi}{4} \qquad \text{or} \qquad x = \frac{3\pi}{4}.$$

Consequently, f is concave up on

$$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$$

and concave down on

$$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right).$$

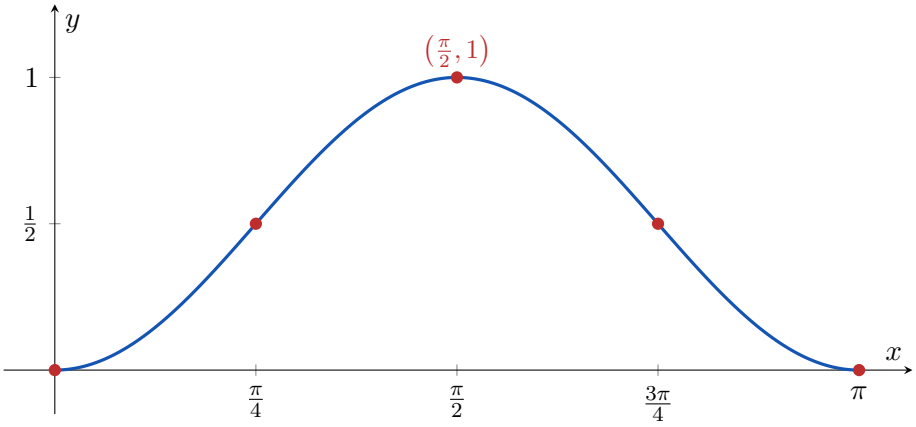
The concavity changes at both values, and

$$f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{1}{2}.$$

Therefore, the inflection points are

$$\boxed{\left(\frac{\pi}{4}, \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{3\pi}{4}, \frac{1}{2}\right).}$$

(e) The graph meets the x-axis at $(0, 0)$ and $(\pi, 0)$. Together with the preceding information, this gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = x^4.$$

Then f is twice differentiable and

$$f''(x) = 12x^2,$$

so $f''(0) = 0$. However, $f''(x) > 0$ for every $x \neq 0$. Thus, f is concave up on both $(-\infty, 0)$ and $(0, +\infty)$, and the concavity does not change at 0. Consequently,

$$(0, f(0)) = (0, 0)$$

is not an inflection point. The condition $f''(c) = 0$ only identifies a possible inflection point; a change in concavity must still be verified.

Calculus 1
Midterm 1

Version E

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} e^x, & x \leqslant 0, \\ ax + b, & 0 < x < 1, \\ 2\ln x + 2, & x \geqslant 1. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 1$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. Consider the equation $x^3 + x = 1$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(0, 1)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = x\sqrt{4 - x^2}$ on the closed interval $[-2, 2]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(-2, 2)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[-2, 2]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[-2, 2]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If f is continuous on $[a, b]$ and attains its absolute maximum at a point $c \in (a, b)$, then $f'(c) = 0$.

Midterm 1 – Solutions

Version E

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 1. At $x = 0$,

$$f(0) = 1, \qquad \lim_{x \to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 1.$$

At $x = 1$,

$$\lim_{x \to 1^-} f(x) = a + b, \qquad f(1) = 2 \ln 1 + 2 = 2.$$

Therefore,

$$a + b = 2.$$

Using $b = 1$, we obtain

$$\boxed{a = 1, \qquad b = 1.}$$

(b) For these values,

$$f(x) = \begin{cases} e^x, & x \leqslant 0, \\ x + 1, & 0 < x < 1, \\ 2 \ln x + 2, & x \geqslant 1. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = e^0 = 1, \qquad f'_+(0) = 1.$$

They agree, so f is differentiable at 0 and $f'(0) = 1$.

At $x = 1$,

$$f'_-(1) = 1, \qquad f'_+(1) = \frac{2}{x}\bigg|_{x=1} = 2.$$

The one-sided derivatives are different, so f is not differentiable at 1.

(c) Since $f(0) = 1$ and $f'(0) = 1$, the tangent line is

$$y - 1 = x - 0,$$

and hence

$$\boxed{y = x + 1.}$$

Problem 2.

(a) Define

$$g(x) = x^3 + x - 1.$$

The function g is continuous on $\mathbb{R}$ and, in particular, on $[0,1]$. Moreover,

$$g(0) = -1 < 0, \qquad g(1) = 1 > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (0,1)$ such that $g(c) = 0$. Hence, $c^3 + c = 1$, so the original equation has at least one solution in $(0,1)$.

(b) At the endpoints of the smaller interval $(1/2, 3/4)$, we have

$$g\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} - 1 = -\frac{3}{8} < 0, \qquad g\left(\frac{3}{4}\right) = \frac{27}{64} + \frac{3}{4} - 1 = \frac{11}{64} > 0.$$

Since g is continuous on $[1/2, 3/4]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(1/2, 3/4).}$$

(c) We have

$$g'(x) = 3x^2 + 1 > 0$$

for every $x \in \mathbb{R}$. Hence, g is strictly increasing on $\mathbb{R}$ and can vanish at most once. Together with part **(a)**, this proves that the solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{2(2 - x^2)}{\sqrt{4 - x^2}}.$$

The denominator is positive on $(-2, 2)$, so $f'(x) = 0$ precisely when $x^2 = 2$. Hence, the critical numbers are

$$\boxed{x = -\sqrt{2} \quad \text{and} \quad x = \sqrt{2}.}$$

(b) The sign chart for f' is

x	$(-2, -\sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2}, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2}, 2)$
$f'(x)$	$-$	0	$+$	0	$-$
f	decreasing		increasing		decreasing

Therefore,

$$\boxed{(-\sqrt{2}, -2) \text{ is a local minimum point,}}$$

and

$$\boxed{(\sqrt{2}, 2) \text{ is a local maximum point.}}$$

(c) We compare the critical numbers and the endpoints:

x	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2
$f(x)$	0	-2	0	2	0

Thus,

$$\boxed{\max_{[-2,2]} f = 2 \text{ at } x = \sqrt{2},} \qquad \boxed{\min_{[-2,2]} f = -2 \text{ at } x = -\sqrt{2}.}$$

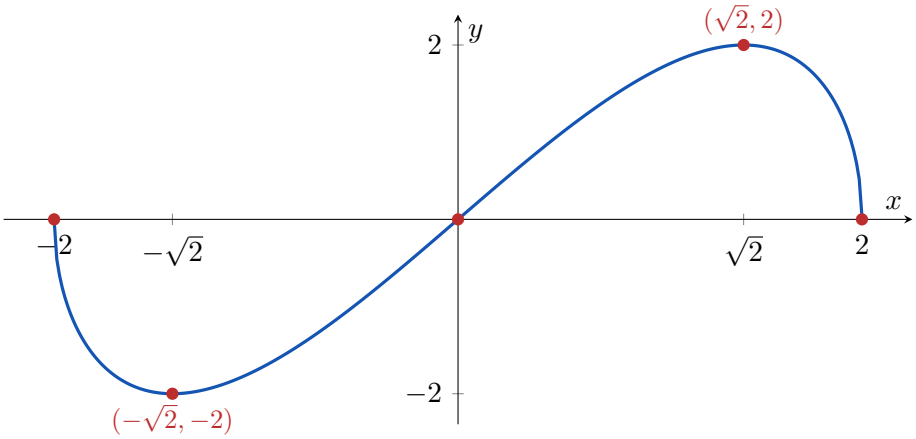
(d) Differentiating again gives

$$f''(x) = \frac{2x(x^2-6)}{(4-x^2)^{3/2}}.$$

For $-2 < x < 2$, we have $x^2 - 6 < 0$. Therefore, $f''(x) > 0$ when $x < 0$ and $f''(x) < 0$ when $x > 0$. Consequently, f is concave up on $(-2, 0)$ and concave down on $(0, 2)$. The concavity changes at 0, and $f(0) = 0$, so the unique inflection point is

$$\boxed{(0, 0).}$$

(e) The graph meets both coordinate axes at $(0, 0)$ and also meets the x-axis at $(-2, 0)$ and $(2, 0)$. Together with the preceding information, this gives the following sketch.



Problem 4.

The statement is **false**. Consider the continuous function

$$f(x) = -|x|, \qquad -1 \leqslant x \leqslant 1.$$

Since

$$f(x) \leqslant 0 = f(0)$$

for every $x \in [-1, 1]$, the function attains its absolute maximum at the interior point $c = 0$. However,

$$f'_-(0) = 1, \qquad f'_+(0) = -1.$$

The one-sided derivatives are different, so $f'(0)$ does not exist. Fermat's theorem gives $f'(c) = 0$ at an interior extremum only when f is differentiable at c.

Calculus 1
Midterm 1

Version F

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} \cos x, & x \leqslant 0, \\ ax^2 + b, & 0 < x < 2, \\ \sqrt{x-1} + 4, & x \geqslant 2. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 2$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. For $x > 0$, consider the equation

$$\frac{1}{x} = x - 1.$$

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(1, 2)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = x + \dfrac{4}{x}$ on the closed interval $[1, 8]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(1, 8)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[1, 8]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[1, 8]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

Every differentiable function $f : \mathbb{R} \to \mathbb{R}$ that is bounded above and below has a point $c \in \mathbb{R}$ such that $f'(c) = 0$.

Midterm 1 – Solutions

Version F

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 2. At $x = 0$,

$$f(0) = \cos 0 = 1, \qquad \lim_{x\to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 1.$$

At $x = 2$,

$$\lim_{x\to 2^-} f(x) = 4a + b, \qquad f(2) = \sqrt{2-1} + 4 = 5.$$

Therefore,

$$4a + b = 5.$$

Using $b = 1$, we obtain

$$\boxed{a = 1, \qquad b = 1.}$$

(b) For these values,

$$f(x) = \begin{cases} \cos x, & x \leqslant 0, \\ x^2 + 1, & 0 < x < 2, \\ \sqrt{x-1} + 4, & x \geqslant 2. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = -\sin 0 = 0, \qquad f'_+(0) = 2x|_{x=0} = 0.$$

They agree, so f is differentiable at 0 and $f'(0) = 0$.

At $x = 2$,

$$f'_-(2) = 2x|_{x=2} = 4, \qquad f'_+(2) = \frac{1}{2\sqrt{x-1}}\bigg|_{x=2} = \frac{1}{2}.$$

The one-sided derivatives are different, so f is not differentiable at 2.

(c) Since $f(0) = 1$ and $f'(0) = 0$, the tangent line is

$$y - 1 = 0(x - 0),$$

and hence

$$\boxed{y = 1.}$$

Problem 2.

(a) Define

$$g(x) = \frac{1}{x} - x + 1.$$

The function g is continuous on $(0, +\infty)$ and, in particular, on $[1, 2]$. Moreover,

$$g(1) = 1 > 0, \qquad g(2) = \frac{1}{2} - 2 + 1 = -\frac{1}{2} < 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (1, 2)$ such that $g(c) = 0$. Hence, $1/c = c - 1$, so the original equation has at least one positive solution.

(b) At the endpoints of the smaller interval $(3/2, 5/3)$, we have

$$g\left(\frac{3}{2}\right) = \frac{2}{3} - \frac{3}{2} + 1 = \frac{1}{6} > 0, \qquad g\left(\frac{5}{3}\right) = \frac{3}{5} - \frac{5}{3} + 1 = -\frac{1}{15} < 0.$$

Since g is continuous on $[3/2, 5/3]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(3/2, 5/3).}$$

(c) We have

$$g'(x) = -\frac{1}{x^2} - 1 < 0$$

for every $x > 0$. Hence, g is strictly decreasing on $(0, +\infty)$ and can vanish at most once there. Together with part **(a)**, this proves that the positive solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = 1 - \frac{4}{x^2} = \frac{(x-2)(x+2)}{x^2}.$$

On $(1, 8)$, the derivative vanishes precisely when $x = 2$. Hence, the only critical number is

$$\boxed{x = 2.}$$

(b) The sign chart for f' is

x	$(1,2)$	2	$(2,8)$
$f'(x)$	$-$	0	$+$
f	decreasing		increasing

Therefore,

$$\boxed{(2, 4) \text{ is a local minimum point.}}$$

There is no local maximum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	1	2	8
$f(x)$	5	4	$\frac{17}{2}$

Thus,

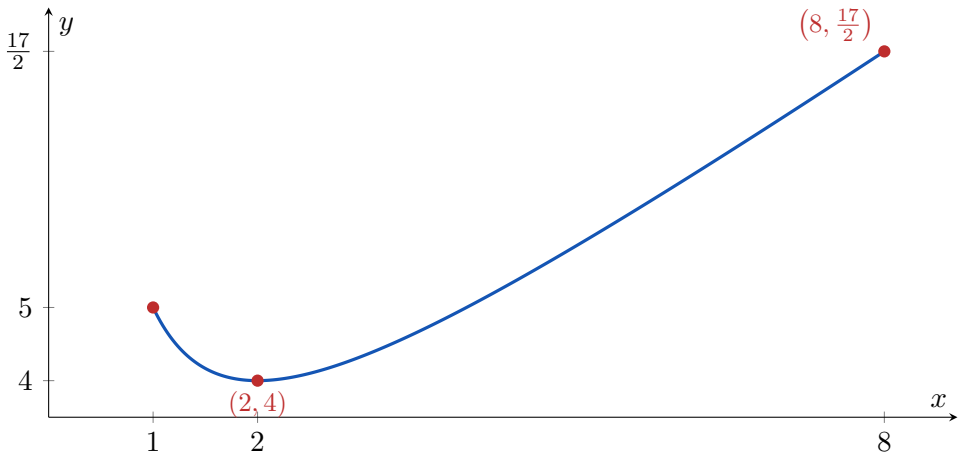
$$\boxed{\max_{[1,8]} f = \frac{17}{2} \text{ at } x = 8,} \qquad \boxed{\min_{[1,8]} f = 4 \text{ at } x = 2.}$$

(d) Differentiating again gives

$$f''(x) = \frac{8}{x^3}.$$

Since $f''(x) > 0$ on $(1, 8)$, the function is concave up throughout $(1, 8)$. There is no inflection point in the interval.

(e) The graph decreases from $(1, 5)$ to the minimum point $(2, 4)$ and then increases to $(8, 17/2)$. It is concave up throughout the interval. This gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = \arctan x.$$

This function is differentiable on $\mathbb{R}$ and bounded because

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$$

for every $x \in \mathbb{R}$. However,

$$f'(x) = \frac{1}{1 + x^2} > 0$$

for every $x \in \mathbb{R}$. Thus, $f'(c) = 0$ has no solution. Boundedness on the whole real line does not guarantee that a function attains an absolute extremum.

Calculus 1
Midterm 1

Version G

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} \sin x + 2, & x \leqslant 0, \\ ax + b, & 0 < x < 2, \\ (x-2)^2 + 4, & x \geqslant 2. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 2$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. For $x > 0$, consider the equation

$$\sqrt{x} = 3 - x.$$

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(1, 2)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = xe^{-x}$ on the closed interval $[0, 3]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(0, 3)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[0, 3]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[0, 3]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has exactly one critical number, then f is not one-to-one.

Midterm 1 – Solutions

Version G

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 2. At $x = 0$,

$$f(0) = \sin 0 + 2 = 2, \qquad \lim_{x\to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 2.$$

At $x = 2$,

$$\lim_{x\to 2^-} f(x) = 2a + b, \qquad f(2) = (2-2)^2 + 4 = 4.$$

Therefore,

$$2a + b = 4.$$

Using $b = 2$, we obtain

$$\boxed{a = 1, \qquad b = 2.}$$

(b) For these values,

$$f(x) = \begin{cases} \sin x + 2, & x \leqslant 0, \\ x + 2, & 0 < x < 2, \\ (x-2)^2 + 4, & x \geqslant 2. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = \cos 0 = 1, \qquad f'_+(0) = 1.$$

They agree, so f is differentiable at 0 and $f'(0) = 1$.

At $x = 2$,

$$f'_-(2) = 1, \qquad f'_+(2) = 2(x-2)|_{x=2} = 0.$$

The one-sided derivatives are different, so f is not differentiable at 2.

(c) Since $f(0) = 2$ and $f'(0) = 1$, the tangent line is

$$y - 2 = x - 0,$$

and hence

$$\boxed{y = x + 2.}$$

Problem 2.

(a) Define

$$g(x) = \sqrt{x} + x - 3.$$

The function g is continuous on $(0, +\infty)$ and, in particular, on $[1, 2]$. Moreover,

$$g(1) = 1 + 1 - 3 = -1 < 0, \qquad g(2) = \sqrt{2} - 1 > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (1, 2)$ such that $g(c) = 0$. Hence, $\sqrt{c} = 3 - c$, so the original equation has at least one positive solution.

(b) At the endpoints of the smaller interval $(3/2, 7/4)$, we have

$$g\left(\frac{3}{2}\right) = \sqrt{\frac{3}{2}} - \frac{3}{2} < 0, \qquad g\left(\frac{7}{4}\right) = \sqrt{\frac{7}{4}} - \frac{5}{4} > 0.$$

The first inequality follows from $\sqrt{3/2} < 3/2$. For the second, both $\sqrt{7/4}$ and $5/4$ are positive, and

$$\frac{7}{4} > \frac{25}{16} = \left(\frac{5}{4}\right)^2.$$

Since g is continuous on $[3/2, 7/4]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(3/2, 7/4).}$$

(c) We have

$$g'(x) = \frac{1}{2\sqrt{x}} + 1 > 0$$

for every $x > 0$. Hence, g is strictly increasing on $(0, +\infty)$ and can vanish at most once there. Together with part **(a)**, this proves that the positive solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x).$$

Since $e^{-x} > 0$, the derivative vanishes precisely when $x = 1$. Hence, the only critical number is

$$\boxed{x = 1.}$$

(b) The sign chart for f' is

x	$(0,1)$	1	$(1,3)$
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{\left(1, \frac{1}{e}\right) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	0	1	3
$f(x)$	0	$\frac{1}{e}$	$\frac{3}{e^3}$

Since $e^2 > 3$, we have $1/e > 3/e^3$. Thus,

$$\boxed{\max_{[0,3]} f = \frac{1}{e} \text{ at } x = 1,} \qquad \boxed{\min_{[0,3]} f = 0 \text{ at } x = 0.}$$

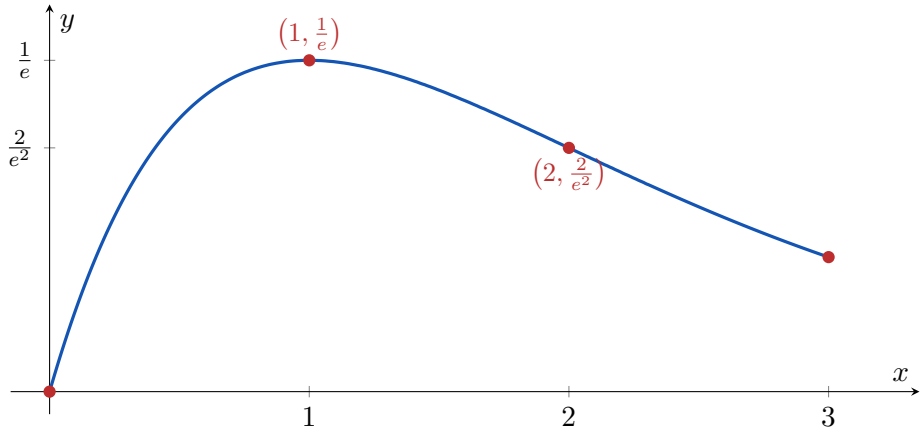
(d) Differentiating again gives

$$f''(x) = e^{-x}(x-2).$$

Since $e^{-x} > 0$, the function is concave down on $(0,2)$ and concave up on $(2,3)$. The concavity changes at $x = 2$, and therefore the unique inflection point is

$$\boxed{\left(2, \frac{2}{e^2}\right).}$$

(e) The graph starts at $(0,0)$, increases to its maximum at $(1, 1/e)$, and then decreases to $(3, 3/e^3)$. Its concavity changes at $(2, 2/e^2)$. This gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = x^3.$$

This function is differentiable on $\mathbb{R}$, and

$$f'(x) = 3x^2.$$

Thus, $x = 0$ is its only critical number. Nevertheless, f is one-to-one: if $f(x_1) = f(x_2)$, then

$$x_1^3 = x_2^3,$$

which implies $x_1 = x_2$. A critical number does not have to be a local extremum.

Calculus 1
Midterm 1

Version H

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} \dfrac{2}{1-x}, & x \leqslant 0, \\ ax + b, & 0 < x < 1, \\ \sqrt{x} + 3, & x \geqslant 1. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 1$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 0$.

Problem 2. For $x > 1$, consider the equation

$$x^2 + \frac{1}{x} = 3.$$

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(1, 2)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = \dfrac{\ln x}{x}$ on the closed interval $[1, e^2]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(1, e^2)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[1, e^2]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[1, e^2]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f'(c) = 0$ and $f''(c) > 0$, then f has an absolute minimum at c.

Midterm 1 – Solutions

Version H

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 1. At $x = 0$,

$$f(0) = \frac{2}{1-0} = 2, \qquad \lim_{x\to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 2.$$

At $x = 1$,

$$\lim_{x\to 1^-} f(x) = a + b, \qquad f(1) = \sqrt{1} + 3 = 4.$$

Therefore,

$$a + b = 4.$$

Using $b = 2$, we obtain

$$\boxed{a = 2, \qquad b = 2.}$$

(b) For these values,

$$f(x) = \begin{cases} \dfrac{2}{1-x}, & x \leqslant 0, \\ 2x + 2, & 0 < x < 1, \\ \sqrt{x} + 3, & x \geqslant 1. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = \left.\frac{2}{(1-x)^2}\right|_{x=0} = 2, \qquad f'_+(0) = 2.$$

They agree, so f is differentiable at 0 and $f'(0) = 2$.

At $x = 1$,

$$f'_-(1) = 2, \qquad f'_+(1) = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = \frac{1}{2}.$$

The one-sided derivatives are different, so f is not differentiable at 1.

(c) Since $f(0) = 2$ and $f'(0) = 2$, the tangent line is

$$y - 2 = 2(x - 0),$$

and hence

$$\boxed{y = 2x + 2.}$$

Problem 2.

(a) Define

$$g(x) = x^2 + \frac{1}{x} - 3.$$

The function g is continuous on $(0, +\infty)$ and, in particular, on $[1, 2]$. Moreover,

$$g(1) = 1 + 1 - 3 = -1 < 0, \qquad g(2) = 4 + \frac{1}{2} - 3 = \frac{3}{2} > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (1, 2)$ such that $g(c) = 0$. Hence, $c^2 + 1/c = 3$, so the original equation has at least one solution with $c > 1$.

(b) At the endpoints of the smaller interval $(3/2, 5/3)$, we have

$$g\left(\frac{3}{2}\right) = \frac{9}{4} + \frac{2}{3} - 3 = -\frac{1}{12} < 0, \qquad g\left(\frac{5}{3}\right) = \frac{25}{9} + \frac{3}{5} - 3 = \frac{17}{45} > 0.$$

Since g is continuous on $[3/2, 5/3]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(3/2, 5/3).}$$

(c) We have

$$g'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} > 0$$

for every $x > 1$. Hence, g is strictly increasing on $(1, +\infty)$ and can vanish at most once there. Together with part **(a)**, this proves that the solution with $x > 1$ is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Since $x^2 > 0$, the derivative vanishes precisely when $\ln x = 1$. Hence, the only critical number is

$$\boxed{x = e.}$$

(b) The sign chart for f' is

x	$(1, e)$	e	(e, e^2)
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{\left(e, \frac{1}{e}\right) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	1	e	e^2
$f(x)$	0	$\frac{1}{e}$	$\frac{2}{e^2}$

Since $e > 2$, we have $1/e > 2/e^2$. Thus,

$$\boxed{\max_{[1,e^2]} f = \frac{1}{e} \text{ at } x = e,} \qquad \boxed{\min_{[1,e^2]} f = 0 \text{ at } x = 1.}$$

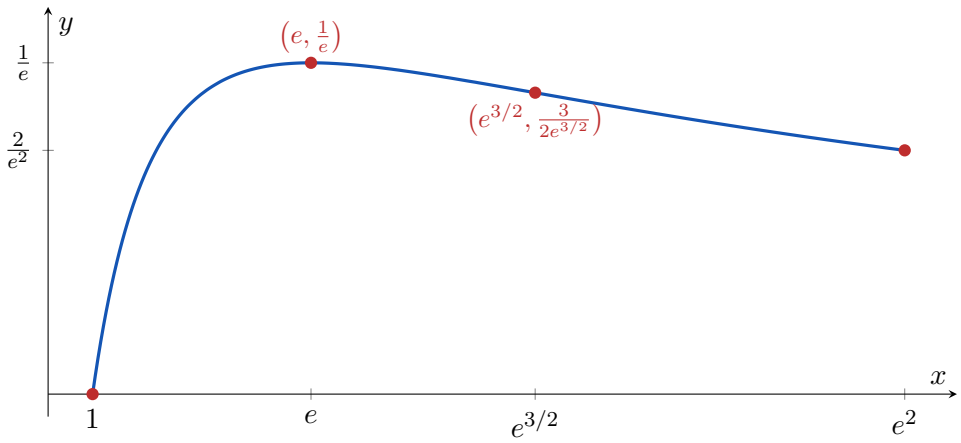
(d) Differentiating again gives

$$f''(x) = \frac{2\ln x - 3}{x^3}.$$

The second derivative vanishes when $\ln x = 3/2$, that is, at $x = e^{3/2}$. Therefore, the function is concave down on $(1, e^{3/2})$ and concave up on $(e^{3/2}, e^2)$. The unique inflection point is

$$\boxed{\left(e^{3/2}, \frac{3}{2e^{3/2}}\right).}$$

(e) The graph starts at $(1, 0)$, increases to its maximum at $(e, 1/e)$, and then decreases to $(e^2, 2/e^2)$. Its concavity changes at $\left(e^{3/2}, 3/(2e^{3/2})\right)$. This gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = x^2 - x^4$$

and take $c = 0$. We have

$$f'(x) = 2x - 4x^3, \qquad f''(x) = 2 - 12x^2.$$

Consequently,

$$f'(0) = 0, \qquad f''(0) = 2 > 0.$$

Thus, 0 is a local minimum. However,

$$f(2) = 4 - 16 = -12 < 0 = f(0),$$

so 0 is not an absolute minimum. The second derivative test gives local, not absolute, information.

Calculus 1
Midterm 1

Version I

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum Score	3	2	4	1	10

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} 2e^{x+1}, & x \leqslant -1, \\ ax + b, & -1 < x < 1, \\ (x-1)^2 + 6, & x \geqslant 1. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = -1$ and at $x = 1$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = -1$.

Problem 2. For $x > 0$, consider the equation $\ln x = 1/x$.

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(1, e)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = xe^{-x/2}$ on the closed interval $[0, 6]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(0, 6)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[0, 6]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[0, 6]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If f is continuous on $[a, b]$, differentiable on (a, b), and $f'(c) = 0$ for some $c \in (a, b)$, then $f(a) = f(b)$.

Midterm 1 – Solutions

Version I

Problem 1.

(a) Continuity only needs to be checked at the junction points -1 and 1. At $x=-1$,

$$f(-1)=2e^0=2, \qquad \lim_{x\to -1^+} f(x)=-a+b.$$

Thus, continuity at -1 requires

$$-a+b=2.$$

At $x=1$,

$$\lim_{x\to 1^-} f(x)=a+b, \qquad f(1)=6.$$

Therefore,

$$a+b=6.$$

Solving these two equations, we obtain

$$\boxed{a=2, \qquad b=4.}$$

(b) For these values,

$$f(x)=\begin{cases} 2e^{x+1}, & x\leqslant -1,\\ 2x+4, & -1<x<1,\\ (x-1)^2+6, & x\geqslant 1. \end{cases}$$

At $x=-1$, the one-sided derivatives are

$$f'_-(-1)=2e^{x+1}\Big|_{x=-1}=2, \qquad f'_+(-1)=2.$$

They agree, so f is differentiable at -1 and $f'(-1)=2$.

At $x=1$,

$$f'_-(1)=2, \qquad f'_+(1)=2(x-1)|_{x=1}=0.$$

The one-sided derivatives are different, so f is not differentiable at 1.

(c) Since $f(-1)=2$ and $f'(-1)=2$, the tangent line is

$$y-2=2(x+1),$$

and hence

$$\boxed{y=2x+4.}$$

Problem 2.

(a) Define

$$g(x) = \ln x - \frac{1}{x}.$$

The function g is continuous on $(0, +\infty)$ and, in particular, on $[1, e]$. Moreover,

$$g(1) = -1 < 0, \qquad g(e) = 1 - \frac{1}{e} > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (1, e)$ such that $g(c) = 0$. Hence, $\ln c = 1/c$, so the original equation has at least one positive solution.

(b) At the endpoints of the smaller interval $(\sqrt{e}, e)$, we have

$$g(\sqrt{e}) = \frac{1}{2} - \frac{1}{\sqrt{e}} < 0, \qquad g(e) = 1 - \frac{1}{e} > 0.$$

Indeed, $\sqrt{e} < 2$, so $1/\sqrt{e} > 1/2$. Since g is continuous on $[\sqrt{e}, e]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(\sqrt{e}, e).}$$

(c) We have

$$g'(x) = \frac{1}{x} + \frac{1}{x^2} > 0$$

for every $x > 0$. Hence, g is strictly increasing on $(0, +\infty)$ and can vanish at most once. Together with part **(a)**, this proves that the positive solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = e^{-x/2}\left(1 - \frac{x}{2}\right).$$

Since $e^{-x/2} > 0$, the derivative vanishes precisely when $x = 2$. Hence, the only critical number in $(0, 6)$ is

$$\boxed{x = 2.}$$

(b) The sign chart for f' is

x	$(0,2)$	2	$(2,6)$
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{\left(2, \frac{2}{e}\right) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	0	2	6
$f(x)$	0	$\frac{2}{e}$	$\frac{6}{e^3}$

Since $e^2 > 3$, we have $2/e > 6/e^3$. Thus,

$$\boxed{\max_{[0,6]} f = \frac{2}{e} \text{ at } x = 2,} \qquad \boxed{\min_{[0,6]} f = 0 \text{ at } x = 0.}$$

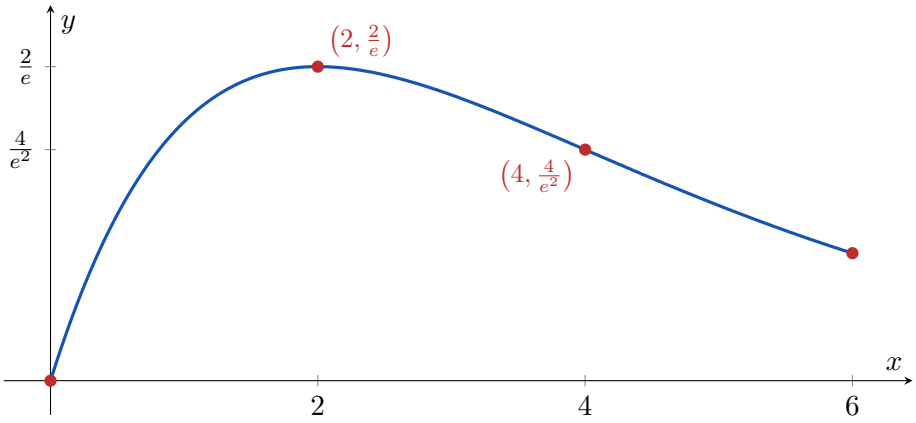
(d) Differentiating again gives

$$f''(x) = \frac{e^{-x/2}}{4}(x-4).$$

Therefore, the function is concave down on $(0,4)$ and concave up on $(4,6)$. Since the concavity changes at $x=4$, the unique inflection point is

$$\boxed{\left(4, \frac{4}{e^2}\right).}$$

(e) Combining the information above gives the following sketch.



Problem 4.

The statement is **false**. Consider $f(x) = x^2$ on $[-1,2]$. This function is continuous on $[-1,2]$ and differentiable on $(-1,2)$. Moreover,

$$f'(0) = 0,$$

but

$$f(-1) = 1 \neq 4 = f(2).$$

Thus, the existence of a point where f' vanishes does not imply equality of the endpoint values. Rolle's theorem gives the implication in the opposite direction.

Calculus 1
Midterm 1

Version J

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} \dfrac{1}{1+x^2}, & x \leqslant 0, \\ ax + b, & 0 < x < 2, \\ \ln(x-1) + 3, & x \geqslant 2. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 2$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 2$.

Problem 2. For $x > 0$, consider the equation

$$\sqrt{x} + \ln x = 2.$$

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(1, e)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution is unique.

Problem 3. Let $f(x) = \ln(1 + x^2)$ on the closed interval $[-2, 2]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(-2, 2)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[-2, 2]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[-2, 2]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) < f(b)$, then $f'(x) > 0$ for every $x \in (a, b)$.

Midterm 1 – Solutions

Version J

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 2. At $x = 0$,

$$f(0) = 1, \qquad \lim_{x\to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 1.$$

At $x = 2$,

$$\lim_{x\to 2^-} f(x) = 2a + b, \qquad f(2) = \ln 1 + 3 = 3.$$

Therefore,

$$2a + b = 3.$$

Using $b = 1$, we obtain

$$\boxed{a = 1, \qquad b = 1.}$$

(b) For these values,

$$f(x) = \begin{cases} \dfrac{1}{1+x^2}, & x \leqslant 0, \\ x + 1, & 0 < x < 2, \\ \ln(x-1) + 3, & x \geqslant 2. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = -\frac{2x}{(1+x^2)^2}\bigg|_{x=0} = 0, \qquad f'_+(0) = 1.$$

They are different, so f is not differentiable at 0.

At $x = 2$,

$$f'_-(2) = 1, \qquad f'_+(2) = \frac{1}{x-1}\bigg|_{x=2} = 1.$$

They agree, so f is differentiable at 2 and $f'(2) = 1$.

(c) Since $f(2) = 3$ and $f'(2) = 1$, the tangent line is

$$y - 3 = x - 2,$$

and hence

$$\boxed{y = x + 1.}$$

Problem 2.

(a) Define

$$g(x) = \sqrt{x} + \ln x - 2.$$

The function g is continuous on $(0, +\infty)$ and, in particular, on $[1, e]$. Moreover,

$$g(1) = -1 < 0, \qquad g(e) = \sqrt{e} - 1 > 0.$$

Therefore, the Intermediate Value Theorem gives a number $c \in (1, e)$ such that $g(c) = 0$. Hence, $\sqrt{c} + \ln c = 2$, so the original equation has at least one positive solution.

(b) At the endpoints of the smaller interval $(1, 9/4)$, we have

$$g(1) = -1 < 0, \qquad g\left(\frac{9}{4}\right) = \ln\left(\frac{9}{4}\right) - \frac{1}{2} > 0.$$

Since $e > 5/2 > 9/4$, this interval is contained in $(1, e)$. Moreover, $e < 3 < 81/16$, so $\sqrt{e} < 9/4$ and therefore $\ln(9/4) > \ln(\sqrt{e}) = 1/2$. Since g is continuous on $[1, 9/4]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(1, 9/4).}$$

(c) We have

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{x} > 0$$

for every $x > 0$. Hence, g is strictly increasing on $(0, +\infty)$ and can vanish at most once. Together with part **(a)**, this proves that the positive solution is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{2x}{1 + x^2}.$$

The denominator is positive, so the derivative vanishes precisely when $x = 0$. Hence, the only critical number in $(-2, 2)$ is

$$\boxed{x = 0.}$$

(b) The sign chart for f' is

x	$(-2, 0)$	0	$(0, 2)$
$f'(x)$	$-$	0	$+$
f	decreasing		increasing

Therefore,

$$\boxed{(0, 0) \text{ is a local minimum point.}}$$

There is no local maximum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	-2	0	2
$f(x)$	$\ln 5$	0	$\ln 5$

Thus,

$$\boxed{\max_{[-2,2]} f = \ln 5 \text{ at } x = -2 \text{ and } x = 2,} \qquad \boxed{\min_{[-2,2]} f = 0 \text{ at } x = 0.}$$

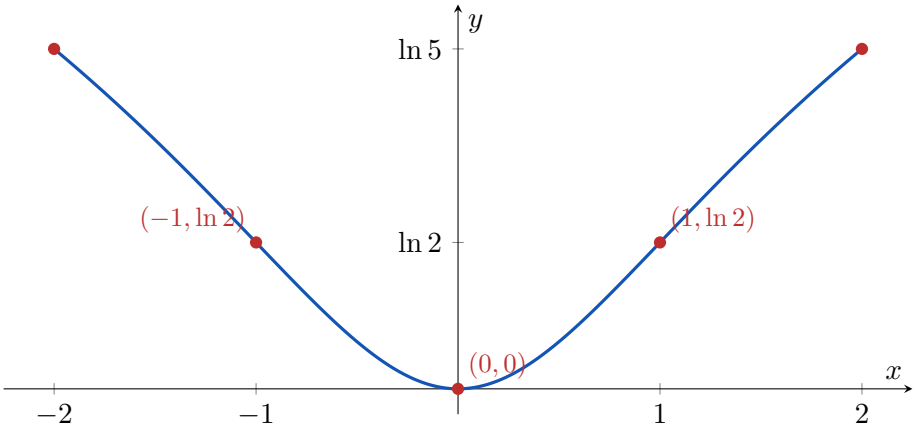
(d) Differentiating again gives

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Therefore, the function is concave down on $(-2,-1)$ and $(1,2)$, and concave up on $(-1,1)$. The concavity changes at $x=-1$ and $x=1$, so the inflection points are

$$\boxed{(-1, \ln 2) \quad \text{and} \quad (1, \ln 2).}$$

(e) Combining the information above gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = x + \sin(2\pi x)$$

on $[0,1]$. This function is continuous on $[0,1]$ and differentiable on $(0,1)$. Moreover,

$$f(0) = 0 < 1 = f(1).$$

However,

$$f'(x) = 1 + 2\pi\cos(2\pi x),$$

and therefore

$$f'\left(\frac{1}{2}\right) = 1 - 2\pi < 0.$$

Thus, a function may have a larger value at the right endpoint without being increasing throughout the interval. The Mean Value Theorem guarantees a point where the derivative equals the average slope, not that the derivative is positive everywhere.

Calculus 1
Midterm 1

Version K

October 27, 2026 – Time: 90 minutes

Problem	1	2	3	4	Total
Maximum	3	2	4	1	10
Score					

Name: _____

Instructions. Write your name in **CAPITAL LETTERS**. Show the main steps and justify your answers. An answer without sufficient reasoning may receive only partial credit. In Problem 4, a false statement must be accompanied by a counterexample.

Problem 1. Consider the function

$$f(x) = \begin{cases} \sqrt{x^2+1}, & x \leqslant 0, \\ ax + b, & 0 < x < 3, \\ \sqrt{x-2} + \dfrac{3}{2}, & x \geqslant 3. \end{cases}$$

(a) **[1.2 pts]** Find $a, b \in \mathbb{R}$ so that f is continuous on $\mathbb{R}$.

(b) **[1.2 pts]** For a and b found in **(a)**, determine whether f is differentiable at $x = 0$ and at $x = 3$.

(c) **[0.6 pts]** Find the equation of the tangent line to the graph of f at $x = 3$.

Problem 2. On the interval $[\pi/2, \pi]$, consider the equation

$$\sin x = \frac{x}{2}.$$

(a) **[0.75 pts]** Prove that the equation has at least one solution in $(\pi/2, \pi)$.

(b) **[0.75 pts]** Find a smaller interval still containing a solution.

(c) **[0.5 pts]** Prove that the solution in $(\pi/2, \pi)$ is unique.

Problem 3. Let $f(x) = \dfrac{x}{(x+1)^2}$ on the closed interval $[0, 4]$.

(a) **[0.7 pts]** Find all critical numbers of f in $(0, 4)$.

(b) **[0.9 pts]** Determine where f is increasing or decreasing and classify its interior local extrema.

(c) **[1.0 pts]** Find the absolute maximum and absolute minimum of f on $[0, 4]$.

(d) **[0.6 pts]** Determine the intervals of concavity and find every inflection point.

(e) **[0.8 pts]** Sketch the graph of f on $[0, 4]$.

Problem 4. **[1 pt]** Decide whether the following statement is true or false and justify your answer.

If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f''(x) > 0$ for every $x \in \mathbb{R}$, then f attains an absolute minimum on $\mathbb{R}$.

Midterm 1 – Solutions

Version K

Problem 1.

(a) Continuity only needs to be checked at the junction points 0 and 3. At $x = 0$,

$$f(0) = \sqrt{1} = 1, \qquad \lim_{x \to 0^+} f(x) = b.$$

Thus, continuity at 0 requires

$$b = 1.$$

At $x = 3$,

$$\lim_{x \to 3^-} f(x) = 3a + b, \qquad f(3) = 1 + \frac{3}{2} = \frac{5}{2}.$$

Therefore,

$$3a + b = \frac{5}{2}.$$

Using $b = 1$, we obtain

$$\boxed{a = \frac{1}{2}, \qquad b = 1.}$$

(b) For these values,

$$f(x) = \begin{cases} \sqrt{x^2 + 1}, & x \leqslant 0, \\ \dfrac{x}{2} + 1, & 0 < x < 3, \\ \sqrt{x - 2} + \dfrac{3}{2}, & x \geqslant 3. \end{cases}$$

At $x = 0$, the one-sided derivatives are

$$f'_-(0) = \left.\frac{x}{\sqrt{x^2 + 1}}\right|_{x=0} = 0, \qquad f'_+(0) = \frac{1}{2}.$$

They are different, so f is not differentiable at 0.

At $x = 3$,

$$f'_-(3) = \frac{1}{2}, \qquad f'_+(3) = \left.\frac{1}{2\sqrt{x - 2}}\right|_{x=3} = \frac{1}{2}.$$

They agree, so f is differentiable at 3 and $f'(3) = 1/2$.

(c) Since $f(3) = 5/2$ and $f'(3) = 1/2$, the tangent line is

$$y - \frac{5}{2} = \frac{1}{2}(x - 3),$$

and hence

$$\boxed{y = \frac{x}{2} + 1.}$$

Problem 2.

(a) Define

$$g(x) = \sin x - \frac{x}{2}.$$

The function g is continuous on $[\pi/2, \pi]$. Moreover,

$$g\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{4} > 0, \qquad g(\pi) = -\frac{\pi}{2} < 0.$$

Since $\pi < 4$, the first value is positive. Therefore, the Intermediate Value Theorem gives a number $c \in (\pi/2, \pi)$ such that $g(c) = 0$. Hence, $\sin c = c/2$.

(b) At the endpoints of the smaller interval $(\pi/2, 2\pi/3)$, we have

$$g\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{4} > 0, \qquad g\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{3} < 0.$$

Indeed, $\pi > 3$ and $\sqrt{3} < 2$, so $2\pi > 3\sqrt{3}$. Since g is continuous on $[\pi/2, 2\pi/3]$, the Intermediate Value Theorem gives a solution in

$$\boxed{(\pi/2, 2\pi/3).}$$

(c) We have

$$g'(x) = \cos x - \frac{1}{2} < 0$$

for every $x \in (\pi/2, \pi)$. Hence, g is strictly decreasing there and can vanish at most once. Together with part **(a)**, this proves that the solution in $(\pi/2, \pi)$ is unique.

Problem 3.

(a) We differentiate:

$$f'(x) = \frac{1 - x}{(x+1)^3}.$$

The denominator is positive on $[0, 4]$, so the derivative vanishes precisely when $x = 1$. Hence, the only critical number in $(0, 4)$ is

$$\boxed{x = 1.}$$

(b) The sign chart for f' is

x	$(0,1)$	1	$(1,4)$
$f'(x)$	$+$	0	$-$
f	increasing		decreasing

Therefore,

$$\boxed{\left(1, \frac{1}{4}\right) \text{ is a local maximum point.}}$$

There is no local minimum point in the interior of the interval.

(c) We compare the critical numbers and the endpoints:

x	0	1	4
$f(x)$	0	$\frac{1}{4}$	$\frac{4}{25}$

Thus,

$$\boxed{\max_{[0,4]} f = \frac{1}{4} \text{ at } x = 1,} \qquad \boxed{\min_{[0,4]} f = 0 \text{ at } x = 0.}$$

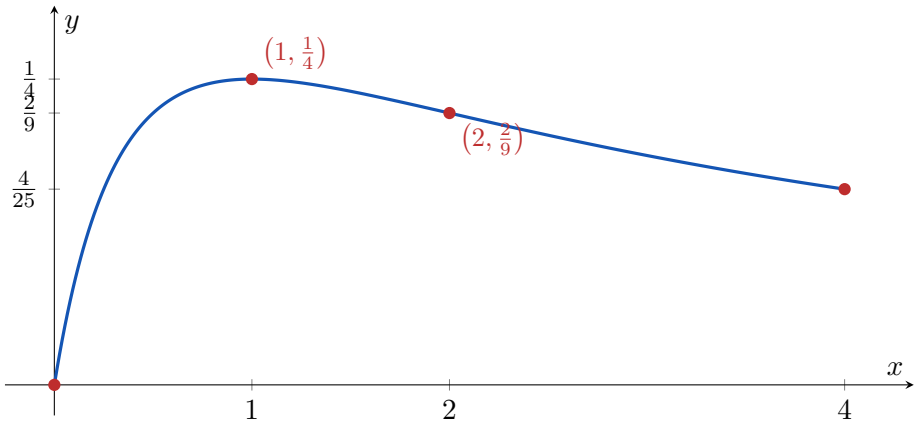
(d) Differentiating again gives

$$f''(x) = \frac{2(x-2)}{(x+1)^4}.$$

Therefore, the function is concave down on $(0, 2)$ and concave up on $(2, 4)$. The concavity changes at $x = 2$, so the unique inflection point is

$$\boxed{\left(2, \frac{2}{9}\right).}$$

(e) Combining the information above gives the following sketch.



Problem 4.

The statement is **false**. Consider

$$f(x) = e^x.$$

This function is twice differentiable on $\mathbb{R}$ and

$$f''(x) = e^x > 0$$

for every $x \in \mathbb{R}$. However, $e^x > 0$ for every x and

$$\lim_{x \to -\infty} e^x = 0.$$

Thus, the infimum of f is 0, but it is not attained. Consequently, f has no absolute minimum on $\mathbb{R}$. Strict convexity alone does not guarantee that an absolute minimum exists.