Calculus 1

Lecture 11: Higher Derivatives and Taylor Polynomials

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What we will be able to do today

- Calculate higher derivatives and recognise recurring patterns.
- Read local information from the values of successive derivatives.
- Construct a Taylor polynomial at any centre a.
- Explain why higher-order polynomials follow a graph more closely near a.
- Build Taylor polynomials for exponential, trigonometric and logarithmic functions.

Higher derivatives

Successive derivatives

Set $f^{(0)} = f$. If $f^{(n-1)}$ is differentiable, then

$$f^{(n)} = \left(f^{(n-1)}\right)'$$

is the n-**th derivative** of f.

We also write

$$f^{(n)}(x) = \frac{d^n f}{dx^n}(x).$$

The notation $f^{(n)}$ denotes a derivative, while f^n denotes a power.

Example: a polynomial eventually disappears

Let

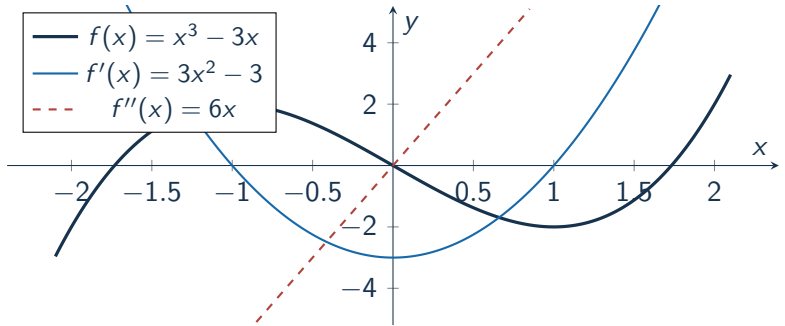
$$f(x) = x^5 - 3x^2 + 2.$$

Then

$$f'(x) = 5x^4 - 6x, \qquad f''(x) = 20x^3 - 6,$$
$$f^{(3)}(x) = 60x^2, \qquad f^{(4)}(x) = 120x,$$
$$f^{(5)}(x) = 120, \qquad f^{(6)}(x) = 0.$$

Every derivative of order greater than the degree of a polynomial is zero.

The first three derivatives describe different features



f' records slope, while f'' records how that slope changes.

The exponential function repeats forever

For $f(x) = e^x$,

$$f'(x) = e^x, \qquad f''(x) = e^x, \qquad f^{(3)}(x) = e^x.$$

Therefore,

$$\boxed{f^{(n)}(x) = e^x \quad \text{for every } n \geq 0.}$$

At the origin, all the derivative values are equal:

$$f^{(n)}(0) = 1.$$

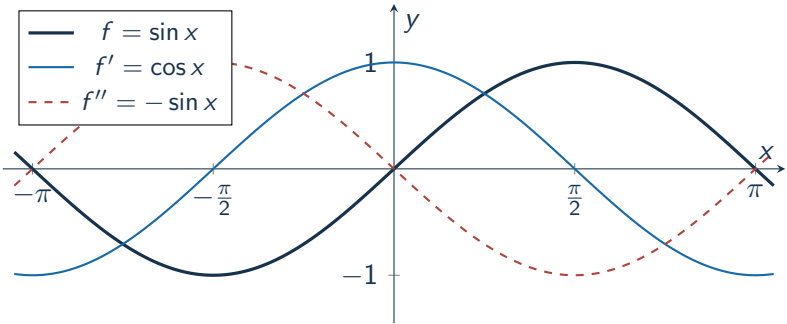
Sine and cosine follow a four-step cycle

Order	0	1	2	3	4
$\sin x$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$\cos x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$

Equivalently,

$$\frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right), \qquad \frac{d^n}{dx^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right).$$

The derivative cycle is a phase shift



After four differentiations, the original graph returns.

Two useful higher-derivative formulas

For $x > 0$ and $n \geq 1$,

$$\boxed{\frac{d^n}{dx^n} \ln x = (-1)^{n-1} \frac{(n-1)!}{x^n}.}$$

For $x \neq 0$ and $n \geq 0$,

$$\boxed{\frac{d^n}{dx^n} \left(\frac{1}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}.}$$

The factorials appear because each differentiation contributes one more factor.

Derivative values form a local fingerprint

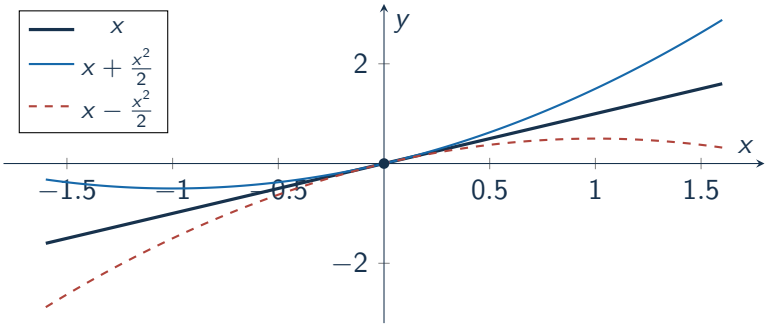
At a fixed point a, the numbers

$$f(a), \quad f'(a), \quad f''(a), \quad f^{(3)}(a), \ldots$$

describe increasingly detailed local behaviour.

- $f(a)$ fixes the point on the graph.
- $f'(a)$ fixes the tangent slope.
- $f''(a)$ records the first change of slope.
- Higher derivatives refine the local shape.

Equal value and slope do not determine the same shape



All three graphs have value 0 and slope 1 at the origin. Their second derivatives differ.

Taylor polynomials

A tangent line already uses derivative data

The tangent line to $y = f(x)$ at $x = a$ is

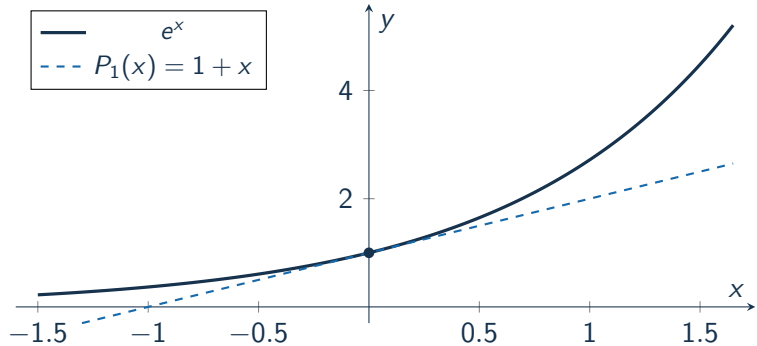
$$L(x) = f(a) + f'(a)(x - a).$$

It satisfies

$$L(a) = f(a), \qquad L'(a) = f'(a).$$

The tangent line is the polynomial that matches the first two pieces of local data. Taylor polynomials continue this idea with higher derivatives.

The tangent line is a first local approximation



Near $x = 0$, the line and the exponential have the same value and slope.

The Taylor polynomial

Suppose that f has derivatives through order n at a. Its **Taylor polynomial through order n centred at a** is

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.}$$

In expanded form,

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin means that the centre is zero

When $a = 0$, the Taylor polynomial becomes the **Maclaurin polynomial**:

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.}$$

The first cases are

$$\begin{aligned} P_0(x) &= f(0), \\ P_1(x) &= f(0) + f'(0)x, \\ P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2. \end{aligned}$$

A reliable construction procedure

To construct P_n centred at a:

1. Calculate $f, f', \ldots, f^{(n)}$.
2. Evaluate each derivative at a.
3. Divide the value $f^{(k)}(a)$ by $k!$.
4. Multiply by $(x - a)^k$.
5. Add all terms from $k = 0$ to $k = n$.

Keep the polynomial in powers of $(x - a)$. This form shows the centre and the coefficients clearly.

Taylor polynomials in graphs

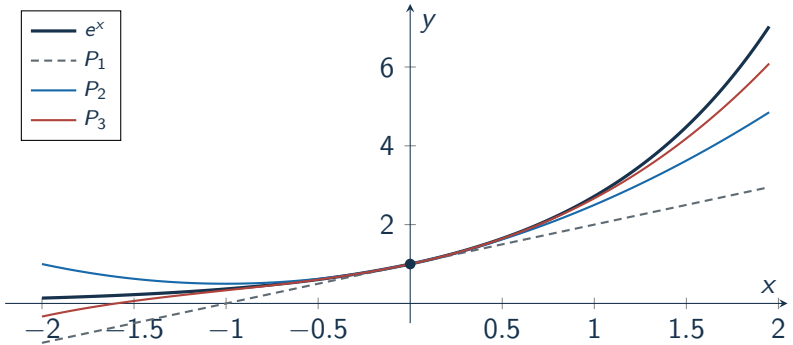
Example: the Maclaurin polynomials of e^x

Since $f^{(k)}(0) = 1$ for every k,

$$
\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= 1 + x, \\
P_2(x) &= 1 + x + \frac{x^2}{2}, \\
P_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.
\end{aligned}
$$

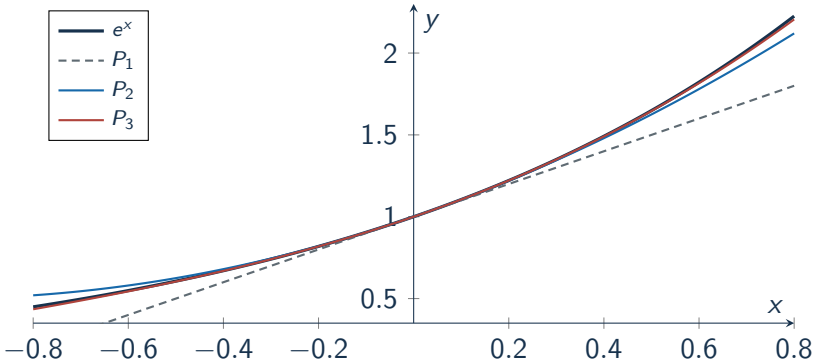
Each new term forces one additional derivative to agree at the origin.

Higher order gives more contact at the centre



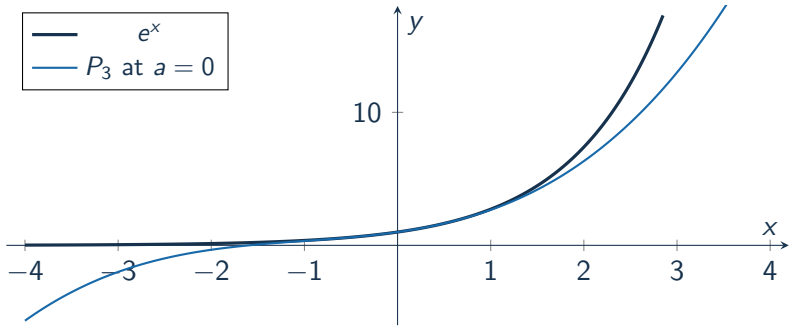
All four graphs meet at $(0, 1)$, but the higher-order polynomials remain close for longer.

A close view makes the matching visible



Near the centre, the graphs become almost indistinguishable.

Taylor approximation is local



Matching several derivatives at one point does not guarantee a good approximation far away.

Example: the Maclaurin polynomial of sine

At $x = 0$, the derivative values are

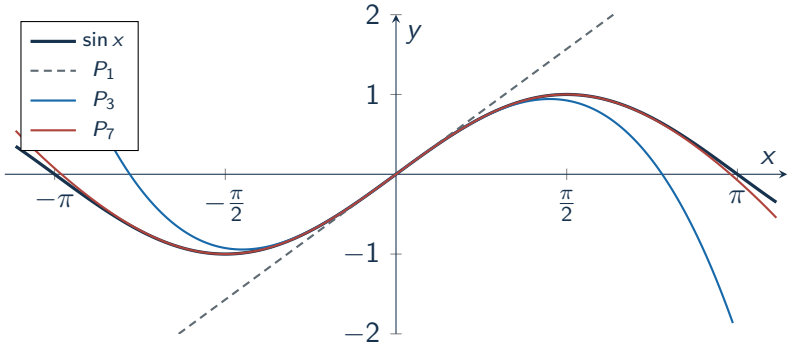
$$0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1, \ldots$$

Therefore,

$$\boxed{P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.}$$

The even coefficients vanish. The polynomial is still called the Taylor polynomial **through order seven**.

The sine polynomials follow a wider interval



The order-seven polynomial captures the oscillating graph well around the origin.

Parity simplifies the calculation

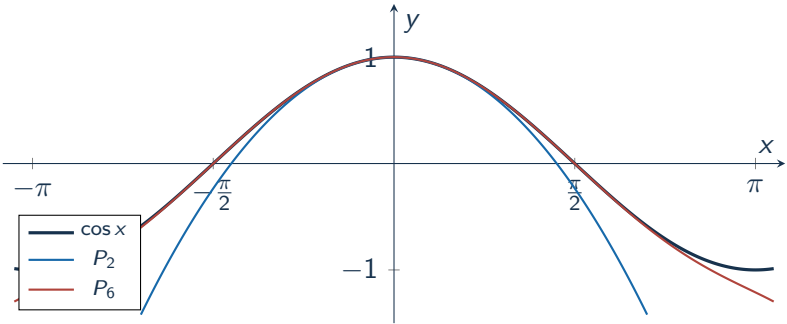
If f is even, then every odd derivative at the origin vanishes. Its Maclaurin polynomial contains only even powers.

If f is odd, then every even derivative at the origin vanishes. Its Maclaurin polynomial contains only odd powers.

Thus,

$$\cos x \rightsquigarrow 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad \sin x \rightsquigarrow x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Cosine produces even polynomials



The symmetry of the function appears directly in the powers of the polynomial.

The centre matters

Example: $\ln x$ centred at $a = 1$

For $f(x) = \ln x$,

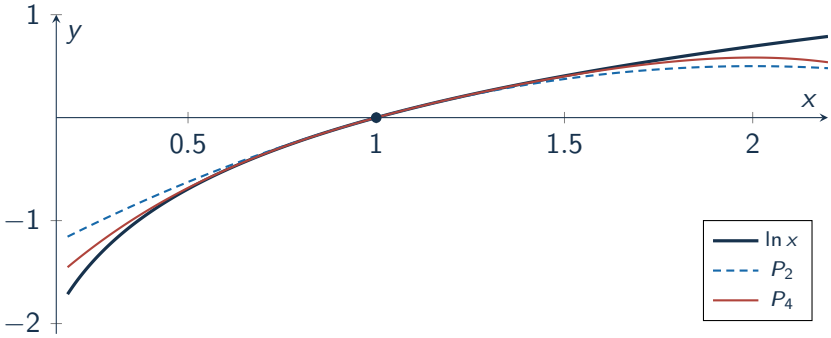
$$f(1) = 0, \qquad f^{(k)}(1) = (-1)^{k-1}(k-1)!, \quad k \geq 1.$$

Therefore,

$$\boxed{P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.}$$

The natural centre is 1, where the logarithm and all its derivatives are easy to evaluate.

The logarithm is approximated around $x = 1$



The approximation deteriorates as the distance from the centre increases.

Example: $\sqrt{x}$ centred at $a = 4$

For $f(x) = \sqrt{x}$,

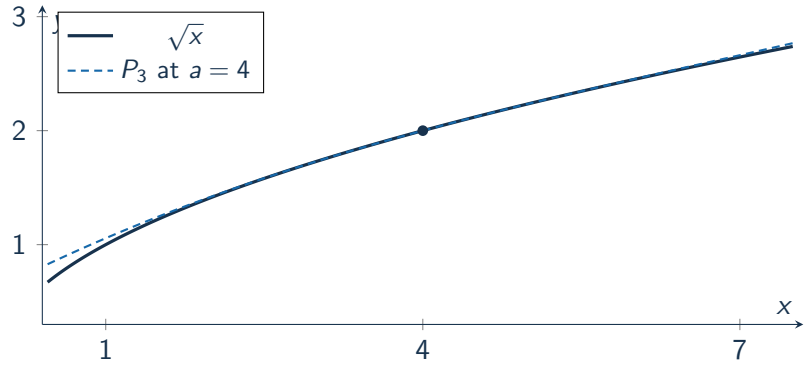
$$f(4) = 2, \quad f'(4) = \frac{1}{4}, \quad f''(4) = -\frac{1}{32}, \quad f^{(3)}(4) = \frac{3}{256}.$$

Hence,

$$\boxed{P_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}.}$$

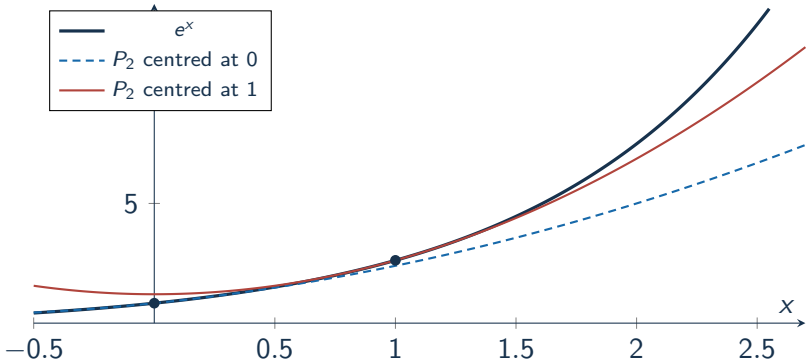
The powers of $(x-4)$ keep the centre visible.

A useful centre makes the graph easier to approximate



The polynomial follows the square-root graph most closely around $x = 4$.

Two centres produce two different polynomials



The polynomial centred at 1 gives the closer local model near $x = 1$.

What a Taylor polynomial does and does not say

It gives a polynomial with the same derivative data as f at one chosen point.

It provides a natural local approximation near that point.

It does not yet provide a numerical error bound.

It does not automatically imply that increasing the order gives a good approximation far from the centre.

The next lecture will study the remainder and rigorous error estimates.

Lecture 11 summary

- Higher derivatives provide increasingly detailed local information.
- P_n matches $f, f', \ldots, f^{(n)}$ at the centre a.
- The coefficient of $(x-a)^k$ is $f^{(k)}(a)/k!$.
- Maclaurin polynomials use the special centre $a = 0$.
- A useful centre lies close to the point where we want an approximation.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

End of Lecture 11

The following solved exercises are extra material.

Extra solved exercises

Exercise 1: find a higher-derivative pattern

Let

$$f(x) = xe^x.$$

Calculate the first four derivatives and conjecture a formula for $f^{(n)}(x)$.

The calculation should reveal the pattern before we write the general formula.

Exercise 1: solution

Using the product rule repeatedly,

$$\begin{aligned} f'(x) &= e^x(x+1), \\ f''(x) &= e^x(x+2), \\ f^{(3)}(x) &= e^x(x+3), \\ f^{(4)}(x) &= e^x(x+4). \end{aligned}$$

Thus,

$$\boxed{f^{(n)}(x) = e^x(x+n).}$$

Indeed, differentiating $e^x(x+n)$ gives $e^x(x+n+1)$, so the pattern continues.

Exercise 2: construct a Maclaurin polynomial

Find the Maclaurin polynomial through order four for

$$f(x) = e^{2x}.$$

Since

$$f^{(k)}(x) = 2^k e^{2x},$$

we have $f^{(k)}(0) = 2^k$.

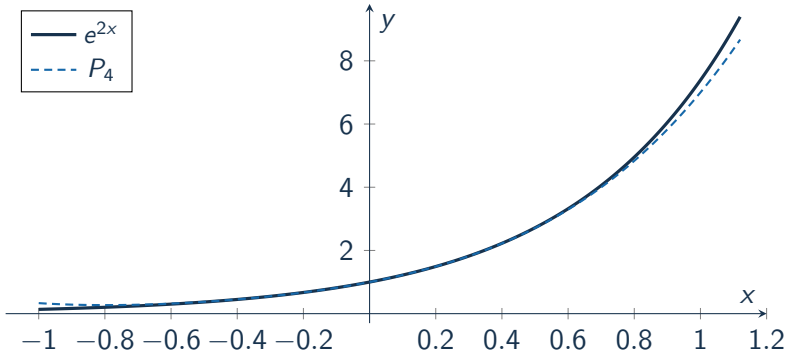
Exercise 2: solution

Substitute the derivative values into the Taylor formula:

$$\begin{aligned} P_4(x) &= 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4. \end{aligned}$$

$$P_4(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

Exercise 2: graphical check



The graph confirms strong agreement near $x = 0$.

Exercise 3: a trigonometric composition

Find the Maclaurin polynomial through order five for

$$f(x) = \sin(2x).$$

The required derivative values are

k	0	1	2	3	4	5
$f^{(k)}(0)$	0	2	0	-8	0	32

Exercise 3: solution

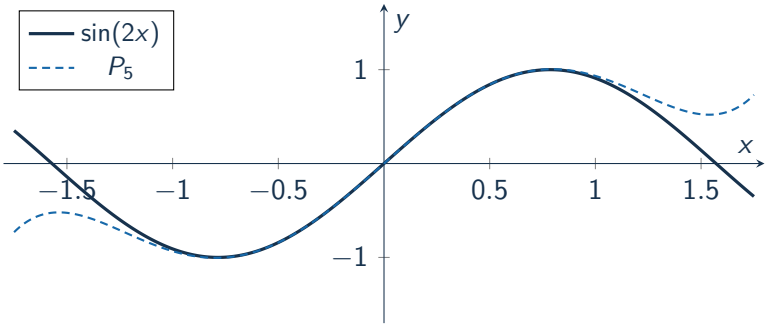
Therefore,

$$\begin{aligned} P_5(x) &= 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 \\ &= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5. \end{aligned}$$

$$P_5(x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$

The answer contains only odd powers because $\sin(2x)$ is odd.

Exercise 3: graphical check



The polynomial captures the steeper oscillation near the origin.

Exercise 4: a centre away from the origin

Find the Taylor polynomial through order three for

$$f(x) = \frac{1}{x}$$

centred at $a = 1$.

Since

$$f^{(k)}(x) = (-1)^k \frac{k!}{x^{k+1}},$$

we have $f^{(k)}(1) = (-1)^k k!$.

Exercise 4: solution

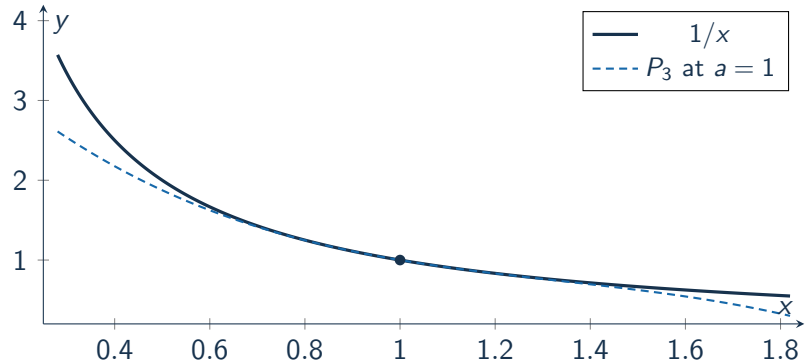
Substituting into the formula gives

$$P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

$$P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$$

The alternating signs come directly from the successive derivatives of $1/x$.

Exercise 4: graphical check



The approximation is most accurate around the centre $x = 1$.

Exercise 5: product and exponential

Find the Maclaurin polynomial through order four for

$$f(x) = x^2 e^x.$$

Successive differentiation gives

$$\begin{aligned} f'(x) &= e^x(x^2 + 2x), \\ f''(x) &= e^x(x^2 + 4x + 2), \\ f^{(3)}(x) &= e^x(x^2 + 6x + 6), \\ f^{(4)}(x) &= e^x(x^2 + 8x + 12). \end{aligned}$$

Exercise 5: solution

At the origin,

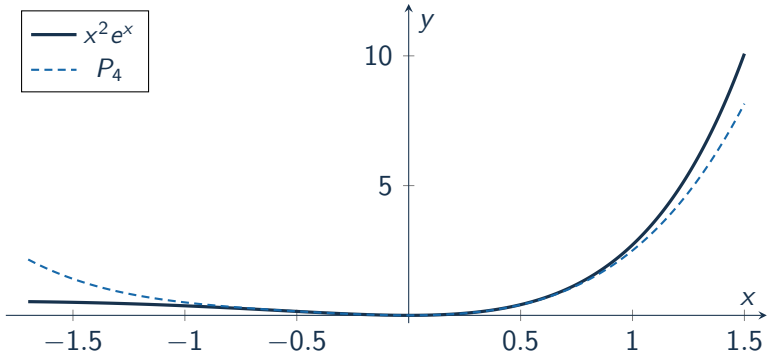
$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = 6, \quad f^{(4)}(0) = 12.$$

Hence,

$$\begin{aligned} P_4(x) &= \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 \\ &= x^2 + x^3 + \frac{x^4}{2}. \end{aligned}$$

$$P_4(x) = x^2 + x^3 + \frac{x^4}{2}$$

Exercise 5: graphical check



Both graphs have a double zero at the origin and share derivatives through order four.

Exercise 6: approximate a logarithm

Use the Taylor polynomial through order four for $\ln x$ centred at 1 to approximate

$$\ln(1.2).$$

Here $x - 1 = 0.2$, so

$$\begin{aligned} P_4(1.2) &= 0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} - \frac{0.2^4}{4} \\ &= 0.182266\overline{6}. \end{aligned}$$

Exercise 6: compare with the function

A calculator gives

$$\ln(1.2) = 0.182321556\ldots$$

while

$$P_4(1.2) = 0.182266666\ldots$$

Therefore,

$$|\ln(1.2) - P_4(1.2)| \approx 5.49 \times 10^{-5}.$$

This numerical comparison shows that the approximation is good. Lecture 12 will provide a rigorous error bound without using the exact value of $\ln(1.2)$.