Calculus 1

Lecture 12: Taylor Remainders and Error Estimates

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What we will be able to do today

- Interpret the remainder as the error in a Taylor approximation.
- Apply Taylor's theorem with the Lagrange remainder.
- Find a rigorous numerical error bound on a specified interval.
- Choose a useful centre and a sufficient Taylor order.

The remainder

Review: the Taylor polynomial

The Taylor polynomial through order n centred at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

It satisfies

$$P_n^{(k)}(a) = f^{(k)}(a), \qquad k = 0, 1, \ldots, n.$$

Derivative matching explains why P_n is a natural local model. It does not yet tell us how far $P_n(x)$ lies from $f(x)$.

The remainder is the exact approximation error

Define

$$\boxed{R_n(x) = f(x) - P_n(x).}$$

Then

$$f(x) = P_n(x) + R_n(x)$$

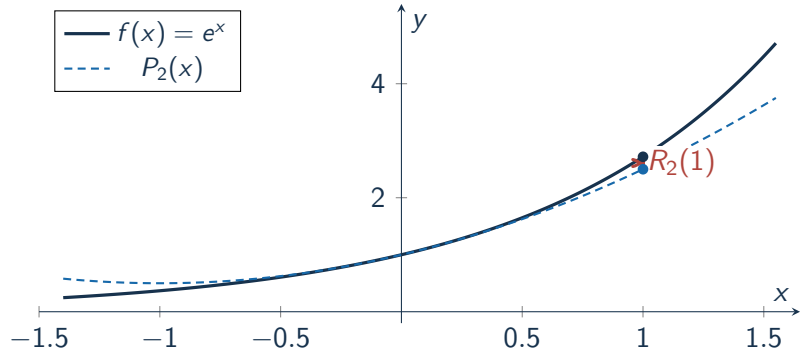
is an exact identity.

The statement

$$f(x) \approx P_n(x)$$

has mathematical content only when we control $|R_n(x)|$.

The remainder is a vertical distance



At a fixed x, the vertical gap between the two graphs has size $|R_n(x)|$.

The Lagrange remainder

Taylor's theorem with Lagrange remainder

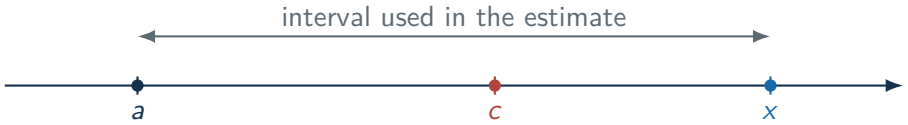
Suppose that f has n continuous derivatives on the closed interval with endpoints a and x, and that $f^{(n+1)}$ exists in its interior.

Then there is a point c strictly between a and x such that

$$\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.}$$

The point c usually remains unknown.

The point c lies between the centre and the target



We bound $f^{(n+1)}$ on the entire interval because the theorem does not identify c.

The Taylor error bound

If

$$\left|f^{(n+1)}(t)\right| \leq M$$

for every t between a and x, then

$$\boxed{|R_n(x)| \leq \frac{M}{(n+1)!}\,|x-a|^{n+1}\,.}$$

Consequently,

$$P_n(x) - E \leq f(x) \leq P_n(x) + E, \qquad E = \frac{M}{(n+1)!}\,|x-a|^{n+1}\,.$$

A reliable error-estimate procedure

1. Choose a centre a close to the target x.
2. Construct the polynomial P_n.
3. Identify the closed interval joining a and x.
4. Calculate $f^{(n+1)}$.
5. Find one valid bound M on the entire interval.

Completing the estimate

After finding M, calculate

$$E = \frac{M}{(n+1)!}\,|x-a|^{n+1}.$$

Then report both the approximation and its guarantee:

$$f(x) \approx P_n(x), \qquad |f(x) - P_n(x)| \le E.$$

Equivalently,

$$\boxed{P_n(x) - E \le f(x) \le P_n(x) + E.}$$

The unknown point c is not an obstacle

Taylor's theorem gives

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

but **normally we cannot calculate** c.

We replace the unknown value by the estimate

$$\left|f^{(n+1)}(c)\right| \leq M.$$

Finding the error

Example: approximate $\sin(0.2)$

The Maclaurin polynomial through order five is

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Therefore,

$$\begin{aligned}\sin(0.2) &\approx P_5(0.2) \\ &= 0.2 - \frac{0.2^3}{6} + \frac{0.2^5}{120} \\ &= 0.1986693333\ldots\end{aligned}$$

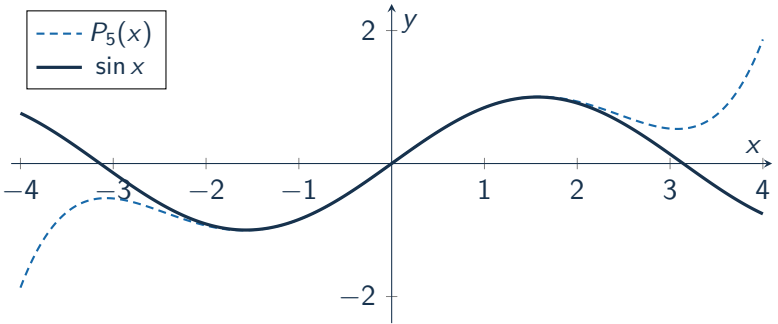
Example: the error in $\sin(0.2)$

The sixth derivative of sine has absolute value at most 1. Hence

$$\begin{aligned} |R_5(0.2)| &\leq \frac{1}{6!}(0.2)^6 \\ &= \frac{0.2^6}{720} \\ &< 8.89 \times 10^{-8}. \end{aligned}$$

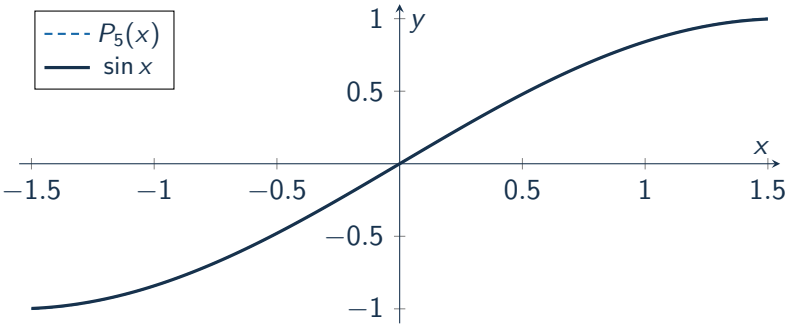
$$\sin(0.2) = 0.1986693333\ldots + R_5, \quad |R_5| < 8.89 \times 10^{-8}$$

A wide view shows the difference



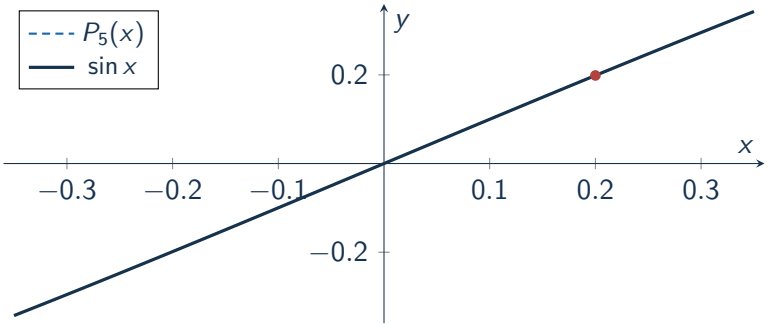
On a large interval, the polynomial and the sine function are clearly different.

The approximation improves near zero



After moving closer to the centre, the two graphs are almost indistinguishable.

Very close to zero, the difference disappears



At $x = 0.2$, the vertical difference cannot be seen at this scale.

Example: approximate $e^{0.1}$

Use the cubic Maclaurin polynomial

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Then

$$P_3(0.1) = 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} = 1.105166666\ldots$$

The approximation uses only addition, multiplication and division.

Example: the error in $e^{0.1}$

On $[0, 0.1]$, the fourth derivative satisfies

$$e^t < 2.$$

Therefore,

$$|R_3(0.1)| \leq \frac{2}{4!}(0.1)^4 < 8.34 \times 10^{-6}.$$

Hence,

$$1.10515833 < e^{0.1} < 1.10517501.$$

Example: a useful centre for $\sqrt{102}$

Take $f(x) = \sqrt{x}$ and centre the polynomial at $a = 100$. Since

$$f(100) = 10, \qquad f'(100) = \frac{1}{20},$$

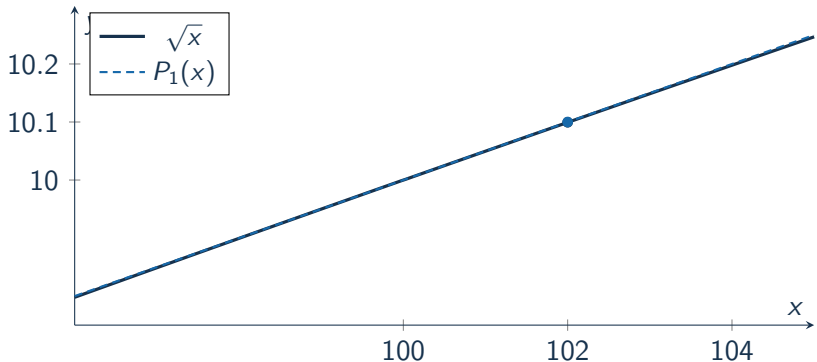
the linear Taylor polynomial is

$$P_1(x) = 10 + \frac{x - 100}{20}.$$

Thus,

$$\sqrt{102} \approx P_1(102) = 10.1.$$

The tangent line is already very accurate



Choosing $a = 100$ keeps the target close to the centre.

Example: the error in $\sqrt{102}$

Since

$$f''(x) = -\frac{1}{4x^{3/2}},$$

we have

$$|f''(x)| \leq \frac{1}{4000} \qquad \text{on } [100, 102].$$

Therefore,

$$|R_1(102)| \leq \frac{1/4000}{2!}(102 - 100)^2 = \frac{1}{2000} = 0.0005.$$

$$\sqrt{102} = 10.1 + R_1, \qquad |R_1| \leq 0.0005$$

Example: approximate ln(1.2)

The Taylor polynomial through order four centred at 1 is

$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

Substitution gives

$$P_4(1.2) = 0.1822666\overline{6}.$$

The fifth derivative is $f^{(5)}(t) = 24/t^5$.

Example: the error in $\ln(1.2)$

On $[1, 1.2]$,

$$\left|f^{(5)}(t)\right| = \frac{24}{t^5} \leq 24.$$

Hence,

$$\begin{aligned} |R_4(1.2)| &\leq \frac{24}{5!}(0.2)^5 \\ &= 6.4 \times 10^{-5}. \end{aligned}$$

$$\ln(1.2) = 0.1822666\overline{6} + R_4, \quad |R_4| \leq 6.4 \times 10^{-5}$$

Choosing the order

Example: approximate $e^{0.5}$ within 10^{-6}

For the Maclaurin polynomial P_n of e^x, use

$$e^t < 2 \qquad \text{on } [0, 0.5].$$

The remainder therefore satisfies

$$|R_n(0.5)| \leq \frac{2}{(n+1)!} \left(\frac{1}{2}\right)^{n+1}.$$

We increase n until this expression becomes smaller than 10^{-6}.

Comparing orders six and seven

For $n = 6$,

$$\frac{2}{7!\,2^7} = \frac{1}{322560} > 10^{-6}.$$

This estimate does not yet provide the required guarantee.

For $n = 7$,

$$\frac{2}{8!\,2^8} = \frac{1}{5160960} < 10^{-6}.$$

The Maclaurin polynomial through order 7 guarantees the required accuracy.

Common mistakes in error estimates

- Bounding $f^{(n)}$ instead of the next derivative $f^{(n+1)}$.
- Using only the value $\left|f^{(n+1)}(a)\right|$ instead of a bound on the interval.
- Forgetting the factorial $(n+1)!$.
- Replacing $|x-a|^{n+1}$ by $|x|^{n+1}$ when $a \neq 0$.
- Reporting an approximation without its error guarantee.

Lecture 12 summary

- $R_n(x) = f(x) - P_n(x)$ is the exact Taylor error.
- The Lagrange form converts the next derivative into a numerical bound.
- The bound must hold on the entire interval between a and x.
- A close centre and a suitable order improve the guarantee.

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

End of Lecture 12

The following solved exercises are extra material.

Extra solved exercises

Exercise 1: approximate cos(0.3)

Use the Maclaurin polynomial through order five:

$$P_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Then

$$\begin{aligned} P_5(0.3) &= 1 - \frac{0.3^2}{2} + \frac{0.3^4}{24} \\ &= 0.9553375. \end{aligned}$$

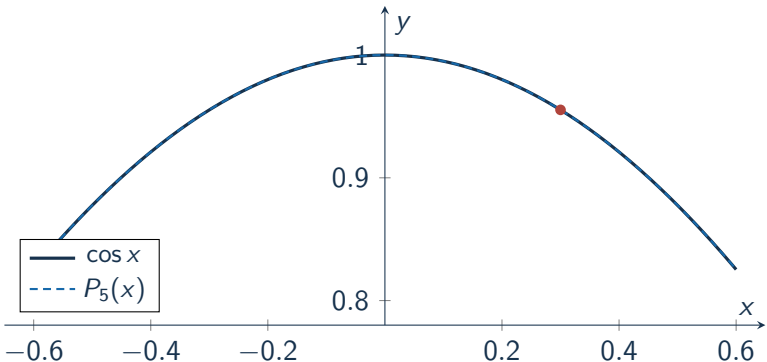
Exercise 1: error estimate

The sixth derivative is $-\cos x$, so $M = 1$ is valid. Therefore,

$$|R_5(0.3)| \leq \frac{0.3^6}{6!} = 1.0125 \times 10^{-6}.$$

$$\cos(0.3) = 0.9553375 + R_5, \quad |R_5| \leq 1.0125 \times 10^{-6}$$

Exercise 1: graph near the centre



The graph and polynomial are indistinguishable near $x = 0.3$ at this scale.

Exercise 2: approximate $\sqrt[3]{9}$

Let $f(x) = x^{1/3}$ and centre at $a = 8$. We have

$$f(8) = 2, \qquad f'(8) = \frac{1}{12}, \qquad f''(8) = -\frac{1}{144}.$$

Thus,

$$P_2(x) = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}.$$

Therefore,

$$P_2(9) = 2 + \frac{1}{12} - \frac{1}{288} = 2.07986111\ldots$$

Exercise 2: error estimate

The third derivative is

$$f^{(3)}(x) = \frac{10}{27x^{8/3}}.$$

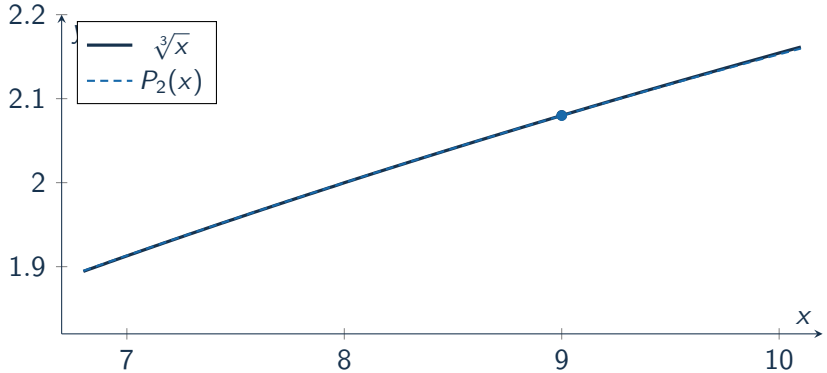
On $[8, 9]$,

$$\left|f^{(3)}(x)\right| \leq \frac{10}{27 \cdot 256} = \frac{5}{3456}.$$

Hence,

$$|R_2(9)| \leq \frac{5/3456}{3!} = \frac{5}{20736} < 2.42 \times 10^{-4}.$$

Exercise 2: graph around $x = 8$



The centre 8 is useful because its cube root is exact.

Exercise 3: approximate $e^{-0.2}$

Using the cubic Maclaurin polynomial,

$$\begin{aligned} P_3(-0.2) &= 1 - 0.2 + \frac{0.2^2}{2} - \frac{0.2^3}{6} \\ &= 0.818666\overline{6}. \end{aligned}$$

On $[-0.2, 0]$, every fourth derivative satisfies $e^t \leq 1$. Hence,

$$|R_3(-0.2)| \leq \frac{0.2^4}{4!} < 6.67 \times 10^{-5}.$$

Exercise 4: choose an order for $\sin(0.7)$

Since every derivative of sine has absolute value at most 1,

$$|R_n(0.7)| \leq \frac{0.7^{n+1}}{(n+1)!}.$$

For $n = 6$,

$$\frac{0.7^7}{7!} > 10^{-5}.$$

For $n = 7$,

$$\frac{0.7^8}{8!} < 1.43 \times 10^{-6} < 10^{-5}.$$

Order 7 guarantees an error smaller than 10^{-5}.

Exercise 5: identify an invalid derivative bound

A student approximates $e^{0.5}$ with $P_3(0.5)$ and writes

$$|R_3(0.5)| \leq \frac{1}{4!}(0.5)^4$$

because $f^{(4)}(0) = 1$.

The argument is invalid. The number M must bound e^t on the whole interval $[0, 0.5]$, while $e^t > 1$ for $t > 0$.

Exercise 5: corrected solution

On $[0, 0.5]$ we may use $e^t < 2$. Therefore,

$$|R_3(0.5)| \leq \frac{2}{4!}(0.5)^4 = \frac{1}{192} \approx 0.00521.$$

The bound is valid, although it may be larger than the actual error.

A convenient bound need not be the smallest possible bound.