

Calculus 1

Week 5: Solutions of Problems 1, 3 and 5

Sheldon Dantas

Faculty of Electrical Engineering, Czech Technical University in Prague

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The same analysis in three settings

The three problems use one common method:

1. determine the domain and the relevant boundary limits;
2. calculate f' and find the critical points;
3. use the sign of f' to obtain the monotonicity intervals;
4. classify the extrema and determine the range;
5. use this information to draw the graph.

For a closed interval, the endpoint values must also be compared when finding absolute extrema.

Problem 1

Problem 1

Find the local and absolute extrema of

$$f(x) = x^3 - 3x^2 - 9x + 5$$

on the interval $[-2, 5]$.

Since f is continuous on the closed bounded interval $[-2, 5]$, the Extreme Value Theorem guarantees that both absolute extrema are attained.

Problem 1

Differentiate and factor:

$$\begin{aligned}f'(x) &= 3x^2 - 6x - 9 \\&= 3(x^2 - 2x - 3) \\&= 3(x + 1)(x - 3).\end{aligned}$$

Therefore,

$$f'(x) = 0 \quad \Longleftrightarrow \quad x = -1 \text{ or } x = 3.$$

Both critical points belong to $[-2, 5]$.

Problem 1

The factors $x + 1$ and $x - 3$ determine the sign of f' .

x	$(-2, -1)$	$(-1, 3)$	$(3, 5)$
$f'(x)$	+	-	+
f	increasing	decreasing	increasing

The derivative changes from positive to negative at $x = -1$, so $x = -1$ gives an interior local maximum.

At $x = 3$, the derivative changes from negative to positive, so $x = 3$ gives an interior local minimum.

Problem 1

For the absolute extrema, compare the function values at the critical points and at both endpoints:

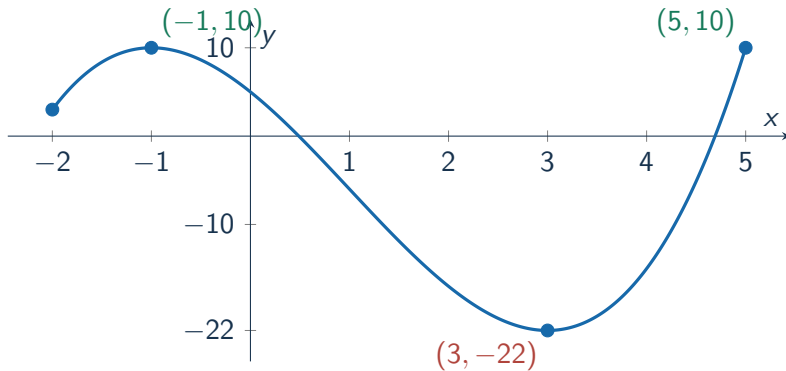
x	-2	-1	3	5
$f(x)$	3	10	-22	10

Thus,

$$f(-1) = f(5) = 10, \quad f(3) = -22.$$

The absolute maximum is 10, attained at $x = -1$ and $x = 5$. The absolute minimum is -22 , attained at $x = 3$.

Problem 1



Problem 3

Problem 3

Carry out a complete graph analysis of

$$f(x) = \frac{x}{x^2 + 1}.$$

Determine the asymptotes, local and absolute extrema and range.

Problem 3

Since $x^2 + 1 > 0$ for every $x \in \mathbb{R}$,

$$\text{Dom}(f) = \mathbb{R}.$$

Also,

$$f(-x) = -f(x),$$

so f is odd. The graph meets both coordinate axes at $(0, 0)$.

At infinity,

$$\lim_{x \rightarrow -\infty} \frac{x}{x^2 + 1} = 0, \quad \lim_{x \rightarrow +\infty} \frac{x}{x^2 + 1} = 0.$$

$y = 0$ is a horizontal asymptote. There is no vertical asymptote.

Problem 3

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} \\ &= \frac{1 - x^2}{(x^2 + 1)^2}. \end{aligned}$$

The denominator is always positive. Hence, the sign of f' depends only on

$$1 - x^2 = (1 - x)(1 + x).$$

The critical points are

$$x = -1 \quad \text{and} \quad x = 1.$$

Problem 3

x	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$f'(x)$	$-$	$+$	$-$
f	decreasing	increasing	decreasing

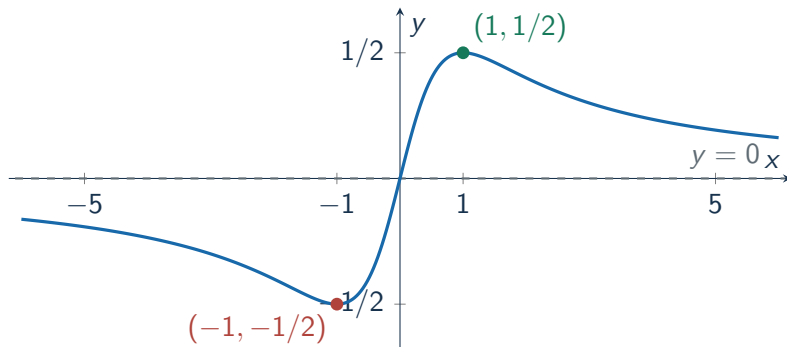
Moreover,

$$f(-1) = -\frac{1}{2}, \quad f(1) = \frac{1}{2}.$$

Together with the limits at infinity, the monotonicity shows that these are also the absolute extrema.

$(-1, -1/2)$ is the absolute minimum, $(1, 1/2)$ is the absolute maximum, and $f(\mathbb{R}) = [-1/2, 1/2]$.

Problem 3



Problem 5

Problem 5

Carry out a complete graph analysis of

$$f(x) = \frac{\ln x}{x}.$$

Determine where the graph meets the coordinate axes, the asymptotes, monotonicity, local and absolute extrema and range.

Problem 5

The logarithm requires $x > 0$, so

$$\text{Dom}(f) = (0, +\infty).$$

There is no point where the graph meets the y -axis. To find where it meets the x -axis, solve

$$\frac{\ln x}{x} = 0 \iff \ln x = 0 \iff x = 1.$$

Thus, the graph meets the x -axis at $(1, 0)$.

Problem 5

At the ends of the domain,

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{x} = -\infty, \quad \lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0.$$

Therefore,

- $x = 0$ is a vertical asymptote;
- $y = 0$ is a horizontal asymptote as $x \rightarrow +\infty$.

Notice that the graph crosses the horizontal asymptote at $(1, 0)$. A graph may cross a horizontal asymptote.

Problem 5

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{x(1/x) - \ln x}{x^2} \\ &= \frac{1 - \ln x}{x^2}. \end{aligned}$$

Since $x^2 > 0$, the sign depends on $1 - \ln x$. The only critical point satisfies

$$1 - \ln x = 0 \quad \Longleftrightarrow \quad \ln x = 1 \quad \Longleftrightarrow \quad x = e.$$

Problem 5

x	$(0, e)$	$(e, +\infty)$
$f'(x)$	+	-
f	increasing	decreasing

At the critical point,

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}.$$

The function is unbounded below because $f(x) \rightarrow -\infty$ as $x \rightarrow 0^+$.

$(e, 1/e)$ is the local and absolute maximum. There is no absolute minimum, and $f((0, +\infty)) = (-\infty, 1/e]$.

Problem 5

