

Practice - 30/9.

① (To warm up). Find the mistake and give the right solution.

a) Calculate $\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|}$.

Possible solution: Since

$$\frac{x^2 - 1}{|x - 1|} = \frac{(x-1)(x+1)}{|x-1|} = x+1,$$

we have that

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1} x+1 = 2.$$

Solution: If $x > 1$, then $|x-1| = x-1$ and

$$\frac{x^2 - 1}{|x - 1|} = \frac{(x-1)(x+1)}{x-1} = x+1$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|} = 2.$$

But if $x < 1$, then $|x-1| = -(x-1)$ and

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1} -(x+1) = -2.$$

← does not exist
← then

b) Calculate $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x} + x)$.



Possible solution: Notice that

$$\sqrt{x^2 + 4x} = \sqrt{x^2 \cdot \left(1 + \frac{4}{x}\right)} = x \sqrt{1 + \frac{4}{x}}$$

and then

$$\sqrt{x^2 + 4x} + x = x \left(\sqrt{1 + \frac{4}{x}} + 1 \right).$$

So,

$$\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 4x} + x) =$$

$$\lim_{x \rightarrow -\infty} x \left(\sqrt{1 + \frac{4}{x}} + 1 \right) = -\infty.$$

Incorrect! We should take care of the absolute value carefully: since $x \rightarrow -\infty$, we have $|x| = -x$ (notice that above we've used $\sqrt{x^2} = x$ and it should be $\sqrt{x^2} = |x| = -x$).

Solution: We can do the following:

$$\sqrt{x^2+4x} + x = \frac{(\sqrt{x^2+4x} + x)(\sqrt{x^2+4x} - x)}{\sqrt{x^2+4x} - x}$$

Is x^2+4x positive for very "small" negative values of x ? Yes!

$$x^2+4x = x(x+4)$$

$$= \frac{x^2+4x-x^2}{\sqrt{x^2+4x} - x} = \frac{4x}{\sqrt{x^2+4x} - x}$$

$$= \frac{(-x) \cdot (-4)}{-x \left(\sqrt{1+\frac{4}{x}} + 1 \right)}$$

$$= \frac{-4}{\sqrt{1+\frac{4}{x}} + 1} \Rightarrow \lim_{x \rightarrow 0^-} \sqrt{x^2+4x} + x = \lim_{x \rightarrow 0^-} \frac{-4}{\sqrt{1+\frac{4}{x}} + 1}$$

$$= -\frac{4}{2} = -2.$$

Be careful with the modulus!!!

c) Find $a, b \in \mathbb{R}$ so that

↑

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x < 0, \\ ax + b, & 0 \leq x < 1, \\ e^{x-1}, & x \geq 1 \end{cases}$$

is continuous on $\mathbb{R}$.

Possible solution: At $x=0$, the formula $ax+b$ is continuous and $f(0)=b$, so f is automatically continuous at $x=0$. At $x=1$, the left-hand limit is $a+b$ while

$$f(1) = e^{1-1} = e^0 = 1.$$

Continuity requires $a+b=1$. ~~Since~~ Consequently, every pair $a, b \in \mathbb{R}$ such that $a+1$ makes f continuous on $\mathbb{R}$.

Solution: We must compare both sides limits!

From the left, we have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin(x)}{x} = 1$$

while

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} ax + b = b.$$

Also, $f(0) = b$. So, $b = 1$. At $x = 1$, we have

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} ax + b = a + b$$

while

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} e^{x-1} = e^0 = 1,$$

i.e., $a + b = 1$. Since $b = 1$, $a = 0$. So,

$\boxed{a=0}$ and $\boxed{b=1}$ make f continuous on

$\mathbb{R}$.

d) let

$$f(x) = x^4 - 5x^2 + 4.$$

Determine whether f has any zeros in $(-3, 3)$.

Possible solution: ~~no~~, it does! ^{not} police that

$$f(-3) = (-3)^4 - 5 \cdot (-3)^2 + 4$$

$$= 81 - 45 + 4 = 40 > 0$$

and $f(3) = +40$. The endpoints values have the same values and, therefore, Bolzano's theorem shows that f has no zeros in $(-3, 3)$.

Possible solution (2): The function f is continuous and even, so it is enough to study it on $[0, 3]$. Since $f(0) = 4 > 0$, ~~and~~

$$f\left(\frac{3}{2}\right) = \frac{81}{16} - \frac{45}{4} + 4 = -\frac{35}{16} < 0$$

and $f(3) = 40 > 0$, Bolzano says there is a ~~zero~~ ^{unique} zero.

in $\left(0, \frac{3}{2}\right)$ and ~~another zero~~ in $\left(\frac{3}{2}, 3\right)$.
a unique zero

Solution: Bolzano guarantees at least one zero between two points where the function has opposite signs. It does not say it is unique!

Notice that

$$\begin{aligned} f(x) &= x^4 - 5x^2 + 4 \\ &= (x^2 - 1)(x^2 - 4) \\ &= (x+1)(x-1)(x-2)(x+2) \end{aligned}$$

→ 4 zeros exactly!

$$\begin{array}{r|l} x^4 - 5x^2 + 4 & x-1 \\ \hline -x^4 + x^3 & \\ \hline x^3 - 5x^2 + 4 & \\ \hline -x^3 + x^2 & \\ \hline -4x^2 + 4 & \\ \hline 4x^2 - 4x & \\ \hline -4x + 4 & \\ \hline 4x - 4 & \\ \hline (0) & \end{array}$$

$$x^4 - 5x^2 + 4 = (x-1)(x^3 + x^2 - 4x - 4)$$

→ Divide $x^3 + x^2 - 4x - 4$ by $x+1$.

② Prove that $\exists!$ a positive real number x such that $\ln(x) + \sqrt{x} = 0$.

Solution: Define $f(x) := \ln(x) + \sqrt{x}$, $x > 0$. Then f is defined on $(0, \infty)$. Consider the interval $[\frac{1}{e}, 1]$. Then,

$$f\left(\frac{1}{e}\right) = \ln(e^{-1}) + \sqrt{\frac{1}{e}} = -1 + e^{-1/2} < 0$$

and

$$f(1) = \ln(1) + \sqrt{1} = 1 > 0.$$

So, $\exists c \in (\frac{1}{e}, 1)$ with $f(c) = 0$, i.e.,

$$\ln(c) + \sqrt{c} = 0.$$

The solution is unique as both functions $\ln(x)$ and $\sqrt{x}$ are strictly increasing.

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