

13 Lecture #9: Week 5

13.1 From derivatives to extrema

In the previous weeks we learned that the derivative measures instantaneous change. We now turn that information into a practical tool. If a quantity is represented by a differentiable function, the sign of the derivative tells us where the quantity is increasing or decreasing, and this is precisely what we need in order to locate its largest and smallest values.

Definition 13.1. Let $f : D \rightarrow \mathbb{R}$ and let $a \in D$.

- The function has an **absolute maximum** at a if $f(x) \leq f(a)$ for every $x \in D$.
- It has an **absolute minimum** at a if $f(a) \leq f(x)$ for every $x \in D$.
- If the corresponding inequality is required only for points x sufficiently close to a , then we speak of a **local maximum** or a **local minimum**.

A maximum or a minimum is also called an **extremum**.

The distinction between local and absolute is important. A hill in the middle of a landscape gives a local maximum of the height, but another hill may be higher. Also, an endpoint can be an absolute extremum without being an interior local extremum in the usual two-sided sense.

We recall two results that organize the search.

Theorem 13.2 (Extreme Value Theorem). If f is continuous on a closed bounded interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$.

Theorem 13.3 (Fermat's theorem). Suppose that c is an interior point of the domain, f is differentiable at c , and f has a local extremum at c . Then $f'(c) = 0$.

The converse of Fermat's theorem is false. For example, $f(x) = x^3$ satisfies $f'(0) = 0$, but it is increasing on both sides of the origin and has no extremum there. Thus, solving $f'(x) = 0$ only produces *candidates*.

Definition 13.4. A number c in the domain of f is a **critical number** if either $f'(c) = 0$ or $f'(c)$ does not exist.

Notice that a point where the function itself is not defined is not a critical number. Such a point may divide the domain into intervals, but it cannot be the location of an extremum of the function.

13.2 Monotonicity and the first derivative test

Theorem 13.5 (Derivative test for monotonicity). Let f be differentiable on an interval I .

- If $f'(x) > 0$ for every $x \in I$, then f is strictly increasing on I .
- If $f'(x) < 0$ for every $x \in I$, then f is strictly decreasing on I .

- If $f'(x) = 0$ for every $x \in I$, then f is constant on I .

This theorem is a direct consequence of the Mean Value Theorem. Indeed, if $x_1 < x_2$, there is $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $x_2 - x_1 > 0$, the sign of the change in f is the sign of $f'(c)$.

Theorem 13.6 (First derivative test). Let c be a critical number and suppose that f is continuous at c .

- If f' changes from positive to negative at c , then f has a local maximum at c .
- If f' changes from negative to positive at c , then f has a local minimum at c .
- If f' has the same sign on both sides of c , then f has no local extremum at c .

Example 13.7. Let

$$f(x) = x^3 - 3x + 1.$$

We have

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

The critical numbers are -1 and 1 . The derivative is positive on $(-\infty, -1)$, negative on $(-1, 1)$, and positive on $(1, +\infty)$. Therefore, f is increasing, then decreasing, then increasing. It has a local maximum at $x = -1$ and a local minimum at $x = 1$, with

$$f(-1) = 3, \quad f(1) = -1.$$

Neither extremum is absolute on $\mathbb{R}$, because the cubic is unbounded above and below.

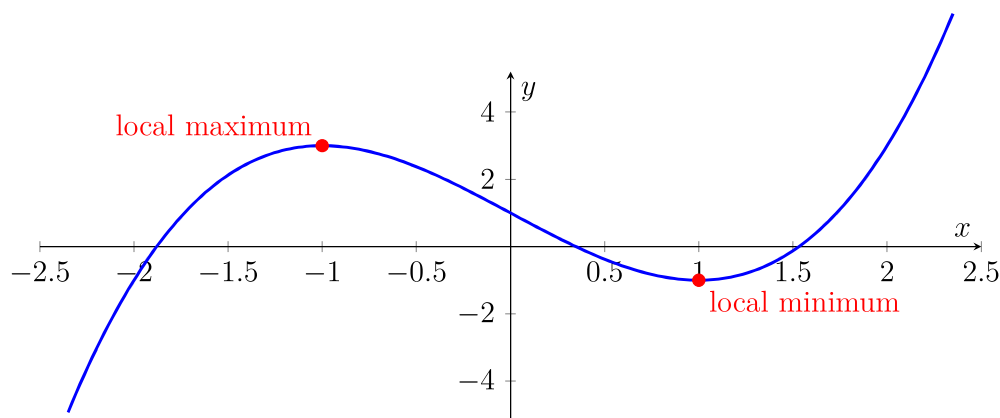


Figure 30: The signs of f' explain the local maximum and minimum of $f(x) = x^3 - 3x + 1$.

13.3 Absolute extrema on an interval

For a continuous function on $[a, b]$, there are only three kinds of candidates: interior critical numbers, the left endpoint, and the right endpoint. We must compare the function values at all of them.

Algorithm 13.8 (Closed-interval method). To find the absolute extrema of a continuous function f on $[a, b]$:

- (1) calculate f' and find every critical number in (a, b) ;
- (2) evaluate f at those critical numbers and at a and b ;
- (3) compare the values. The largest is the absolute maximum and the smallest is the absolute minimum.

Example 13.9. Find the absolute extrema of $f(x) = x^3 - 3x$ on $[-2, 3]$. Since

$$f'(x) = 3(x^2 - 1),$$

the interior critical numbers are -1 and 1 . We evaluate

$$f(-2) = -2, \quad f(-1) = 2, \quad f(1) = -2, \quad f(3) = 18.$$

Thus, the absolute maximum is 18, attained at $x = 3$, and the absolute minimum is -2 , attained at both $x = -2$ and $x = 1$. An absolute extremum need not be unique.

Remark 13.10. If the interval is open or unbounded, the Extreme Value Theorem does not guarantee that extrema exist. For instance, $f(x) = x$ has neither a maximum nor a minimum on $(0, 1)$, even though it is bounded there. Always inspect the domain and its endpoints before starting the derivative calculation.

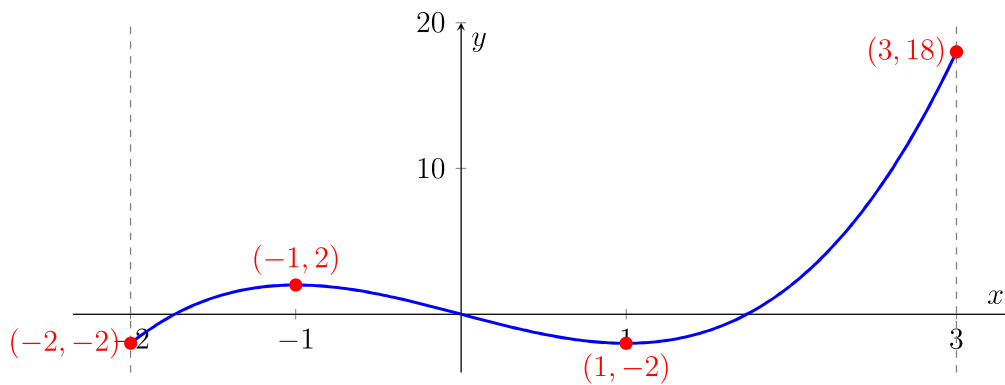


Figure 31: For a closed interval, endpoints and interior critical numbers must all be compared.