

14 Lecture #10: Week 5

14.1 The qualitative meaning of the second derivative

The first derivative records whether the graph rises or falls. The second derivative records how the tangent slopes change. If $f'' > 0$, then f' is increasing and the graph bends upward. If $f'' < 0$, then f' is decreasing and the graph bends downward. This information is geometric: it helps us distinguish the shape of two graphs even when both are increasing or both are decreasing.

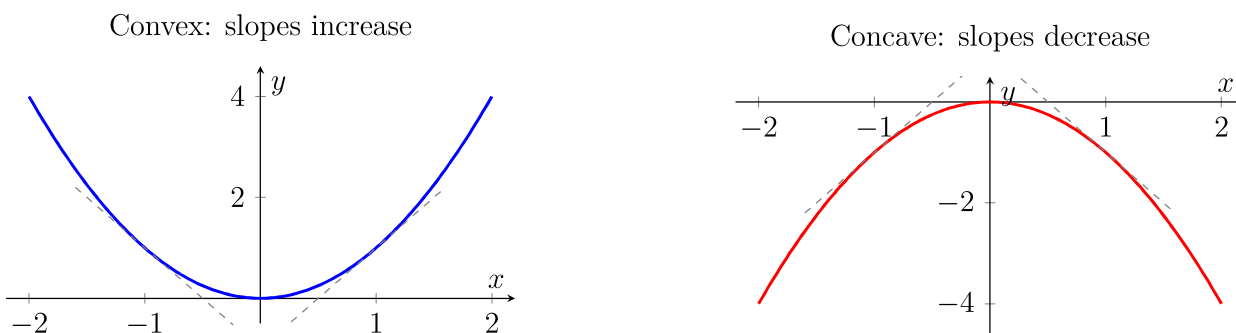


Figure 32: The sign of f'' describes how the tangent slopes change.

14.2 Concavity and convexity

Monotonicity records whether a graph goes up or down. Concavity records how its slope changes. Informally, a convex graph bends upward like a cup, whereas a concave graph bends downward like a cap.

Definition 14.1. Let $f : I \rightarrow \mathbb{R}$, where I is an interval.

- The function is **convex** on I if, for every $x, y \in I$ and $0 \leq t \leq 1$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

- It is **concave** on I if the reverse inequality holds.

For a convex function, the chord joining two points of the graph lies above the graph. For a differentiable convex function, every tangent line lies below the graph. The reverse statements hold for a concave function.

Theorem 14.2 (Second derivative criterion). Suppose that f is twice differentiable on an interval I .

- If $f''(x) > 0$ on I , then f is convex on I .
- If $f''(x) < 0$ on I , then f is concave on I .

The reason is easy to remember. If $f'' > 0$, then f' is increasing, so the tangent slopes become larger as we move from left to right. The graph therefore bends upward.

Definition 14.3. An interior point c is an **inflection point** if the graph changes from convex to concave, or from concave to convex, at c .

A solution of $f''(x) = 0$ is only a candidate for an inflection point. The concavity must actually change. The function x^4 has $f''(0) = 0$ but is convex on both sides of zero, so the origin is not an inflection point. In contrast, x^3 changes from concave to convex at zero.

There is another useful geometric consequence. If f is differentiable and convex on an interval, then the tangent line at any a in the interval lies below the graph:

$$f(x) \geq f(a) + f'(a)(x - a).$$

For a concave function the inequality is reversed. For instance, $f(x) = e^x$ is convex because $f''(x) = e^x > 0$. Its tangent line at 0 is $y = 1 + x$, and therefore

$$e^x \geq 1 + x$$

for every $x \in \mathbb{R}$. Thus, concavity and convexity can also produce useful inequalities without solving a separate optimization problem.

Theorem 14.4 (Second derivative test). Suppose that $f'(c) = 0$ and $f''(c)$ exists.

- If $f''(c) > 0$, then f has a local minimum at c .
- If $f''(c) < 0$, then f has a local maximum at c .
- If $f''(c) = 0$, the test is inconclusive.

Example 14.5. For $f(x) = x^4 - 4x^2$, we have

$$f'(x) = 4x(x^2 - 2), \quad f''(x) = 12x^2 - 8.$$

The critical numbers are 0 and $\pm\sqrt{2}$. Since $f''(0) = -8 < 0$, the origin is a local maximum. Since $f''(\pm\sqrt{2}) = 16 > 0$, the two points with $x = \pm\sqrt{2}$ are local minima. The possible inflection numbers satisfy $12x^2 - 8 = 0$, hence $x = \pm\sqrt{2/3}$. The sign of f'' changes at both, so both are genuine inflection points.

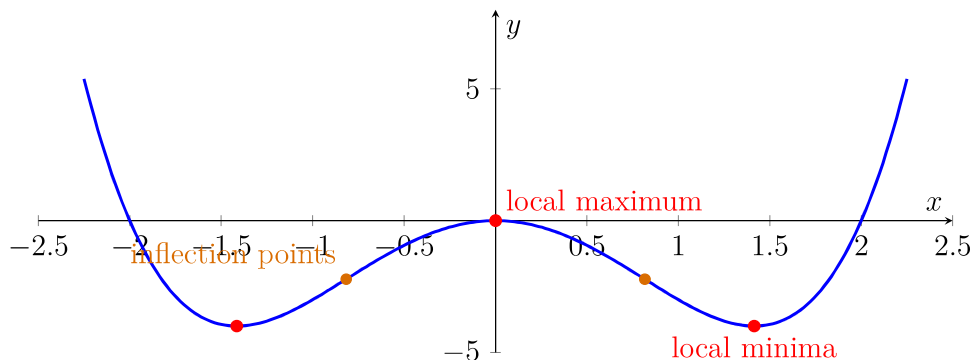


Figure 33: Extrema and inflection points play different roles in the shape of a graph.

14.3 Vertical, horizontal and oblique asymptotes

An asymptote is a line that the graph approaches in a precise limiting sense. It is not a barrier: a graph may cross a horizontal or oblique asymptote.

Definition 14.6. • The line $x = a$ is a **vertical asymptote** if at least one of $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ equals $+\infty$ or $-\infty$.

- The line $y = L$ is a **horizontal asymptote** as $x \rightarrow +\infty$ if

$$\lim_{x \rightarrow +\infty} f(x) = L.$$

The definition at $-\infty$ is analogous.

- The line $y = mx + n$, with $m \neq 0$, is an **oblique asymptote** as $x \rightarrow +\infty$ if

$$\lim_{x \rightarrow +\infty} (f(x) - (mx + n)) = 0.$$

When the relevant limits exist, an oblique asymptote can be found from

$$m = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}, \quad n = \lim_{x \rightarrow \pm\infty} (f(x) - mx).$$

For rational functions, polynomial division is often faster.

Example 14.7. For

$$f(x) = \frac{x^2 + 1}{x - 1},$$

the denominator vanishes at $x = 1$ while the numerator does not, so $x = 1$ is a vertical asymptote. Polynomial division gives

$$f(x) = x + 1 + \frac{2}{x - 1}.$$

Therefore, $y = x + 1$ is an oblique asymptote at both $+\infty$ and $-\infty$.

14.4 A systematic graphing procedure

A graph should be the conclusion of an analysis, not a guess based on a few points. The following order avoids repeated work.

Algorithm 14.8 (Graph analysis). (1) Find the domain and note any symmetry or periodicity.

- (2) Find intercepts and determine signs when useful.
- (3) Calculate limits at boundary points and at infinity; identify asymptotes.
- (4) Use f' to find intervals of increase and decrease and all local extrema.
- (5) Use f'' to find intervals of convexity and concavity and all inflection points.
- (6) Combine the information in a coherent sketch.

Example 14.9 (Complete analysis of a rational function). Consider

$$f(x) = \frac{x^3}{x^2 - 1}.$$

The domain is $\mathbb{R} \setminus \{-1, 1\}$. The function is odd, so its graph is symmetric about the origin. Its only intercept is $(0, 0)$. The denominator gives the vertical asymptotes $x = -1$ and $x = 1$. Polynomial division yields

$$f(x) = x + \frac{x}{x^2 - 1},$$

so $y = x$ is an oblique asymptote at both ends.

For monotonicity,

$$f'(x) = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}.$$

The denominator is positive on the domain. Thus, the sign is controlled by $x^2 - 3$, apart from the zero at $x = 0$. The function is increasing on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, +\infty)$, and decreasing on each domain interval between $-\sqrt{3}$ and $\sqrt{3}$. Hence,

$$f(-\sqrt{3}) = -\frac{3\sqrt{3}}{2}$$

is a local maximum and

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{2}$$

is a local minimum. Although $f'(0) = 0$, the sign does not change there.

The second derivative is

$$f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Since $x^2 + 3 > 0$, a sign chart shows that f is concave on $(-\infty, -1)$ and $(0, 1)$, and convex on $(-1, 0)$ and $(1, +\infty)$. The concavity changes at $x = 0$, so $(0, 0)$ is an inflection point. It does not make sense to call $x = \pm 1$ inflection points because the function is not defined there.

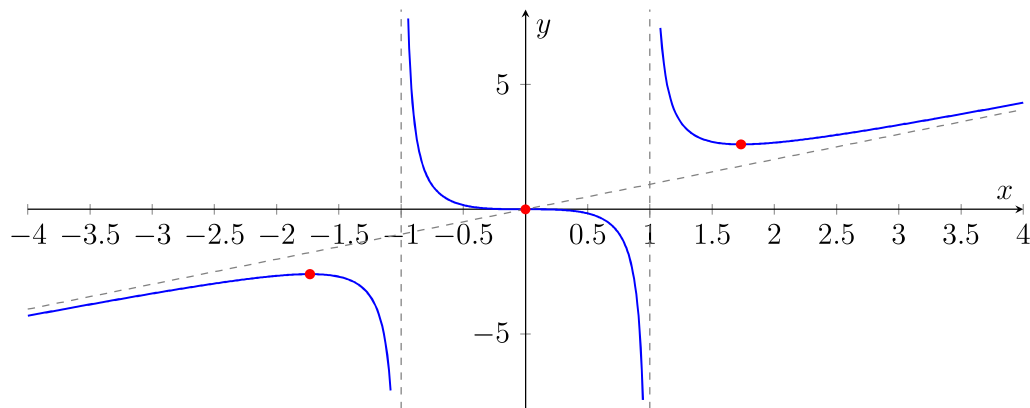


Figure 34: The graph of $x^3/(x^2 - 1)$ together with its three asymptotes.