17 Lecture #11: Week 7

17.1 Higher derivatives

The first derivative measures an instantaneous rate of change. If f' is itself differentiable, we can differentiate again. Repeating this process produces the higher derivatives of f.

Definition 17.1. We write $f^{(0)} = f$. If $f^{(n-1)}$ is differentiable, then

$$f^{(n)} = \left(f^{(n-1)}\right)'$$

is the **n-th derivative** of f. In particular,

$$f^{(1)} = f', \qquad f^{(2)} = f'', \qquad f^{(3)} = f'''.$$

The parentheses in $f^{(n)}$ are important. The symbol $f^{(n)}$ denotes a derivative, whereas f^n normally denotes a power of the function. Other common notations are

$$\frac{d^n f}{dx^n} \qquad \text{and} \qquad \frac{d^n y}{dx^n},$$

when $y = f(x)$.

The first few derivatives describe different aspects of the graph. The value $f(a)$ gives the height, $f'(a)$ gives the tangent slope, and $f''(a)$ gives information about the local concavity. Higher derivatives record further local changes.

Example 17.2 (A polynomial eventually disappears)**.** Let

$$f(x) = x^5 - 3x^2 + 2.$$

Differentiating one step at a time gives

$$\begin{aligned} f'(x) &= 5x^4 - 6x, \\ f''(x) &= 20x^3 - 6, \\ f^{(3)}(x) &= 60x^2, \\ f^{(4)}(x) &= 120x, \\ f^{(5)}(x) &= 120, \\ f^{(6)}(x) &= 0. \end{aligned}$$

Every derivative after the sixth is also zero. More generally, repeated differentiation eventually annihilates every polynomial.

The exponential function behaves differently:

$$f(x) = e^x, \qquad f'(x) = e^x, \qquad f''(x) = e^x, \qquad f^{(3)}(x) = e^x.$$

Every derivative is again e^x.

For sine and cosine, the derivatives repeat in a cycle of length four:

$$\begin{aligned} &\sin x \longmapsto \cos x \longmapsto -\sin x \longmapsto -\cos x \longmapsto \sin x, \\ &\cos x \longmapsto -\sin x \longmapsto -\cos x \longmapsto \sin x \longmapsto \cos x. \end{aligned}$$

Example 17.3 (A higher-derivative pattern)**.** Let $f(x) = xe^x$. We calculate

$$\begin{aligned} f'(x) &= e^x(x+1), \\ f''(x) &= e^x(x+2), \\ f^{(3)}(x) &= e^x(x+3), \\ f^{(4)}(x) &= e^x(x+4). \end{aligned}$$

The pattern suggests

$$f^{(n)}(x) = e^x(x+n).$$

Indeed, differentiating $e^x(x+n)$ gives $e^x(x+n+1)$, so the pattern continues.

17.2 From the tangent line to the Taylor polynomial

The tangent line to $y = f(x)$ at $x = a$ is

$$P_1(x) = f(a) + f'(a)(x-a).$$

It agrees with f in value and first derivative at the centre a. If more derivatives are known, we can construct a polynomial that matches more of the local behaviour of f.

Consider

$$P_2(x) = A + B(x-a) + C(x-a)^2.$$

The conditions

$$P_2(a) = f(a), \qquad P_2'(a) = f'(a), \qquad P_2''(a) = f''(a)$$

give

$$A = f(a), \qquad B = f'(a), \qquad C = \frac{f''(a)}{2!}.$$

The same idea produces the general formula.

Definition 17.4. Suppose that f has derivatives through order n at a. The **Taylor polynomial through order n centred at a** is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

When $a = 0$, it is called the **Maclaurin polynomial**:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

By construction,

$$P_n^{(k)}(a) = f^{(k)}(a), \qquad k = 0, 1, \ldots, n.$$

Thus the derivative values at the centre act as a local fingerprint. The polynomial has order n, but its actual degree may be smaller if some of the highest coefficients vanish.

The following procedure is reliable.

(1) Calculate the derivatives from f through $f^{(n)}$.

(2) Evaluate each derivative at the centre a.

(3) Divide the value $f^{(k)}(a)$ by $k!$.

(4) Multiply by $(x-a)^k$ and add the resulting terms.

(5) When $a \neq 0$, keep the answer in powers of $x-a$.

Example 17.5 (The fourth-order polynomial for e^{2x})**.** Let $f(x) = e^{2x}$ and take $a = 0$. The derivatives needed are

$$\begin{aligned} f(x) &= e^{2x}, \\ f'(x) &= 2e^{2x}, \\ f''(x) &= 4e^{2x}, \\ f^{(3)}(x) &= 8e^{2x}, \\ f^{(4)}(x) &= 16e^{2x}. \end{aligned}$$

At the origin, the corresponding values are $1, 2, 4, 8, 16$. Therefore,

$$\begin{aligned} P_4(x) &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4. \end{aligned}$$

Example 17.6 (The fifth-order polynomial for $\sin(2x)$)**.** Let $f(x) = \sin(2x)$. Differentiating one step at a time,

$$\begin{aligned} f(x) &= \sin(2x), \\ f'(x) &= 2\cos(2x), \\ f''(x) &= -4\sin(2x), \\ f^{(3)}(x) &= -8\cos(2x), \\ f^{(4)}(x) &= 16\sin(2x), \\ f^{(5)}(x) &= 32\cos(2x). \end{aligned}$$

At $x = 0$, the values are $0, 2, 0, -8, 0, 32$. Hence,

$$P_5(x) = 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5.$$

Only odd powers appear, as expected because $\sin(2x)$ is odd.

17.3 Taylor polynomials and their graphs

Taylor polynomials of higher order have more contact with the graph at the centre. This does not mean that the approximation is equally good everywhere. A Taylor polynomial is a local model, and the graphs may separate when we move farther from the centre.

For $f(x) = e^x$ centred at 0, the first polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= 1 + x, \\ P_2(x) &= 1 + x + \frac{x^2}{2}, \\ P_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}. \end{aligned}$$

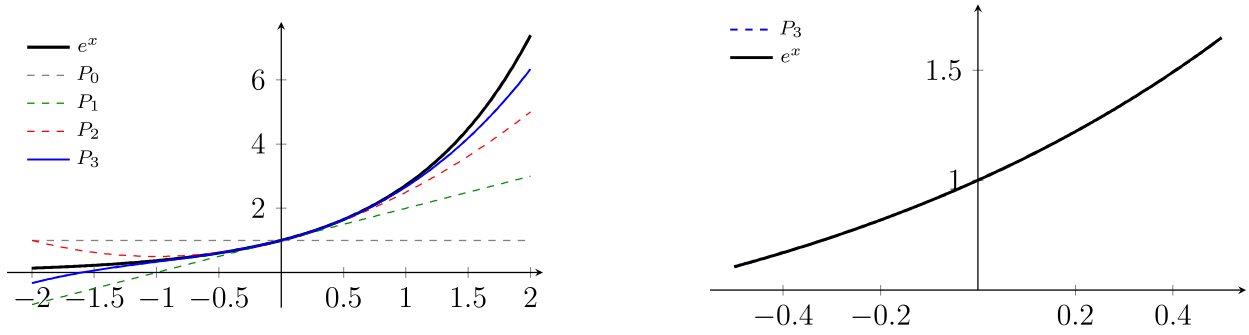


Figure 35: Farther from the centre the Taylor polynomials separate, whereas near 0 the cubic polynomial and e^x are almost indistinguishable.

Parity simplifies Maclaurin polynomials. If f is even, only even powers occur. If f is odd, only odd powers occur. For example,

$$\begin{aligned} \sin x &\approx x - \frac{x^3}{3!} + \frac{x^5}{5!}, \\ \cos x &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}. \end{aligned}$$

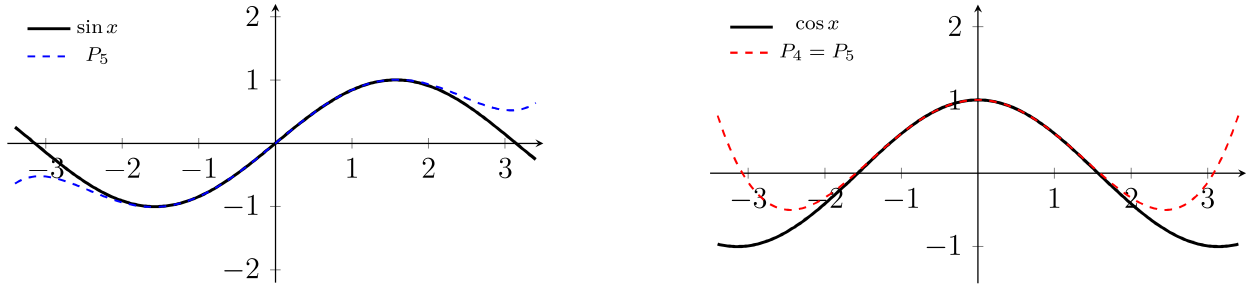


Figure 36: The Maclaurin polynomials of sine and cosine inherit their odd and even symmetry.

17.4 The centre matters

A useful centre is close to the target point and has derivative values that are easy to calculate. The same function produces different Taylor polynomials at different centres.

Example 17.7 ($\ln x$ centred at 1)**.** For $f(x) = \ln x$, we calculate

$$\begin{aligned} f(x) &= \ln x, \\ f'(x) &= \frac{1}{x}, \\ f''(x) &= -\frac{1}{x^2}, \\ f^{(3)}(x) &= \frac{2}{x^3}. \end{aligned}$$

At $a = 1$, these values are $0, 1, -1, 2$. Therefore,

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

The approximation is designed for values of x close to 1.

Example 17.8 ($\sqrt{x}$ centred at 4)**.** Let $f(x) = \sqrt{x}$. The required derivatives are

$$\begin{aligned} f(x) &= \sqrt{x}, \\ f'(x) &= \frac{1}{2\sqrt{x}}, \\ f''(x) &= -\frac{1}{4x^{3/2}}. \end{aligned}$$

At $a = 4$,

$$f(4) = 2, \qquad f'(4) = \frac{1}{4}, \qquad f''(4) = -\frac{1}{32}.$$

Consequently,

$$P_2(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}.$$

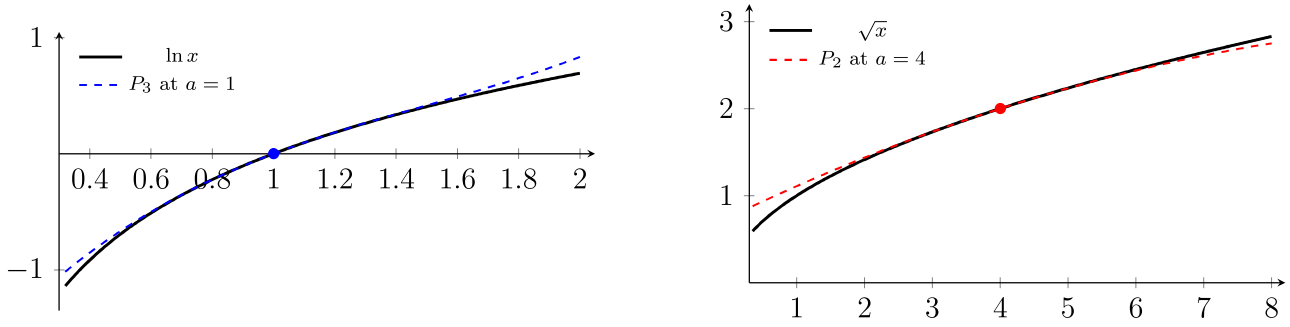


Figure 37: A Taylor polynomial is built around a specified centre.

Example 17.9 (A centre away from the origin)**.** Let $f(x) = 1/x$ and take $a = 1$. We calculate

$$\begin{aligned} f(x) &= \frac{1}{x}, \\ f'(x) &= -\frac{1}{x^2}, \\ f''(x) &= \frac{2}{x^3}, \\ f^{(3)}(x) &= -\frac{6}{x^4}. \end{aligned}$$

At $x = 1$, the values are $1, -1, 2, -6$. Thus,

$$P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3.$$

The Taylor polynomial matches derivative data at one point. By itself, it does not provide a numerical guarantee for the approximation. That guarantee comes from the remainder.