18 Lecture #12: Week 7

18.1 The exact remainder

Let P_n be the Taylor polynomial of f through order n centred at a. The difference between the function and the polynomial is the remainder.

Definition 18.1. The **remainder** after order n is

$$R_n(x) = f(x) - P_n(x).$$

Equivalently,

$$f(x) = P_n(x) + R_n(x).$$

This is an exact identity. An approximation appears only when we discard the remainder. At a fixed value of x, the number $|R_n(x)|$ is the vertical distance between the graph of f and the graph of P_n.

Since the function and its Taylor polynomial have matching derivatives at the centre,

$$R_n(a) = R_n'(a) = \cdots = R_n^{(n)}(a) = 0.$$

The error function is therefore very flat at a, but it normally becomes more visible as we move away from the centre.

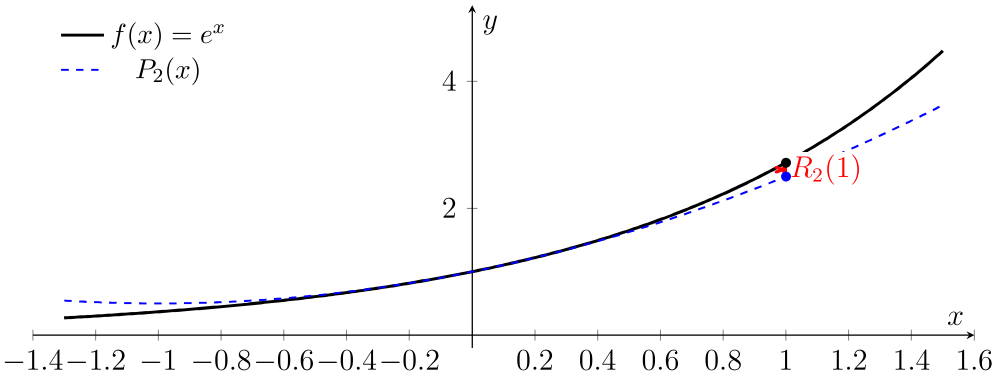


Figure 38: The remainder is the signed vertical difference between $f(x)$ and $P_n(x)$.

18.2 Taylor's theorem with Lagrange remainder

Theorem 18.2 (Taylor's theorem with Lagrange remainder)**.** Suppose that f has n continuous derivatives on the closed interval with endpoints a and x, and that $f^{(n+1)}$ exists on its interior. Then there is a point c strictly between a and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

In other words,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The point c depends on the target x and on the order n. We do not usually know its value. This is not an obstacle: we replace the unknown derivative value at c by a bound that is valid on the entire interval.

Corollary 18.3 (Taylor error bound)**.** If

$$\left|f^{(n+1)}(t)\right| \leqslant M$$

for every t between a and x, then

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}\,|x-a|^{n+1}\,.$$

If

$$E = \frac{M}{(n+1)!}\,|x-a|^{n+1}\,,$$

then the estimate can also be written as the guaranteed interval

$$P_n(x) - E \leqslant f(x) \leqslant P_n(x) + E.$$

The three parts of the bound play different roles. The number M controls the next derivative on the whole interval, the factorial appears in the denominator, and $|x-a|^{n+1}$ measures the distance from the centre. Increasing the order often improves the bound, but not automatically: the value used for M also changes with the derivative order.

To estimate an error reliably, use the following procedure.

(1) Choose a useful centre a close to the target x.

(2) Construct the Taylor polynomial P_n.

(3) Identify the full closed interval joining a and x.

(4) Calculate the next derivative $f^{(n+1)}$.

(5) Find a valid bound M for its absolute value on the whole interval.

(6) Substitute into the Lagrange estimate.

(7) Report both the approximation and its guaranteed error.

18.3 A rigorous error envelope

Example 18.4 (The cosine polynomial through order five)**.** For $f(x) = \cos x$, the derivatives through order six are

$$\begin{aligned}
f(x) &= \cos x,\\
f'(x) &= -\sin x,\\
f''(x) &= -\cos x,\\
f^{(3)}(x) &= \sin x,\\
f^{(4)}(x) &= \cos x,\\
f^{(5)}(x) &= -\sin x,\\
f^{(6)}(x) &= -\cos x.
\end{aligned}$$

At the origin, the polynomial through order five is

$$P_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

The coefficient of x^5 vanishes, so the actual degree is four. Nevertheless, the remainder after order five involves the sixth derivative. Since

$$\left|f^{(6)}(t)\right| = |\cos t| \leqslant 1$$

for every real t, the error satisfies

$$|\cos x - P_5(x)| \leqslant \frac{|x|^6}{6!} = \frac{|x|^6}{720}.$$

Thus the graph of cosine must lie between

$$P_5(x) - \frac{|x|^6}{720} \qquad \text{and} \qquad P_5(x) + \frac{|x|^6}{720}.$$

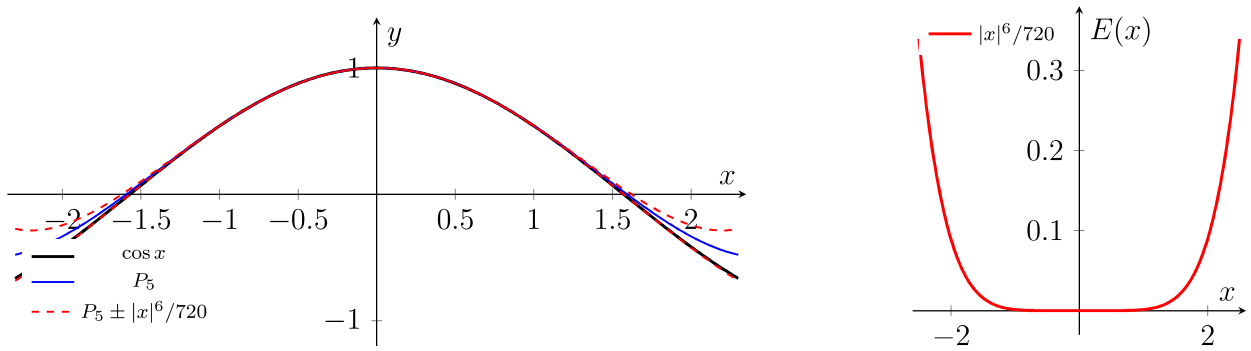


Figure 39: The cosine graph remains inside the Lagrange error envelope. The permitted error grows with the distance from the centre.

18.4 Numerical approximations with guaranteed errors

Example 18.5 (Approximating $\sin(0.2)$)**.** The Maclaurin polynomial through order five is

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Therefore,

$$\begin{aligned} \sin(0.2) &\approx P_5(0.2) \\ &= 0.2 - \frac{0.2^3}{6} + \frac{0.2^5}{120} \\ &= 0.1986693333\ldots \end{aligned}$$

The sixth derivative of sine is $-\sin x$, whose absolute value is at most 1. Consequently,

$$|R_5(0.2)| \leqslant \frac{0.2^6}{6!} < 8.89 \times 10^{-8}.$$

Thus

$$\sin(0.2) = 0.1986693333\ldots + R_5, \qquad |R_5| < 8.89 \times 10^{-8}.$$

Example 18.6 (Approximating $e^{0.1}$)**.** The cubic Maclaurin polynomial is

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Hence,

$$P_3(0.1) = 1.105166666\ldots$$

The fourth derivative of e^x is e^x. On $[0, 0.1]$, we may use $M = 2$. Therefore,

$$|R_3(0.1)| \leqslant \frac{2}{4!}(0.1)^4 < 8.34 \times 10^{-6}.$$

It follows that

$$1.10515833 < e^{0.1} < 1.10517501.$$

Example 18.7 (A useful centre for $\sqrt{102}$)**.** Take $f(x) = \sqrt{x}$ and centre the polynomial at $a = 100$. Since

$$f(100) = 10, \qquad f'(100) = \frac{1}{20},$$

the linear Taylor polynomial is

$$P_1(x) = 10 + \frac{x - 100}{20}.$$

Thus

$$\sqrt{102} \approx P_1(102) = 10.1.$$

The next derivative is

$$f''(x) = -\frac{1}{4x^{3/2}}.$$

On $[100, 102]$,

$$|f''(x)| \leqslant \frac{1}{4 \cdot 100^{3/2}} = \frac{1}{4000}.$$

Therefore,

$$|R_1(102)| \leqslant \frac{1/4000}{2!}(102 - 100)^2 = \frac{1}{2000} = 0.0005.$$

Consequently,

$$\sqrt{102} = 10.1 + R_1, \qquad |R_1| \leqslant 0.0005.$$

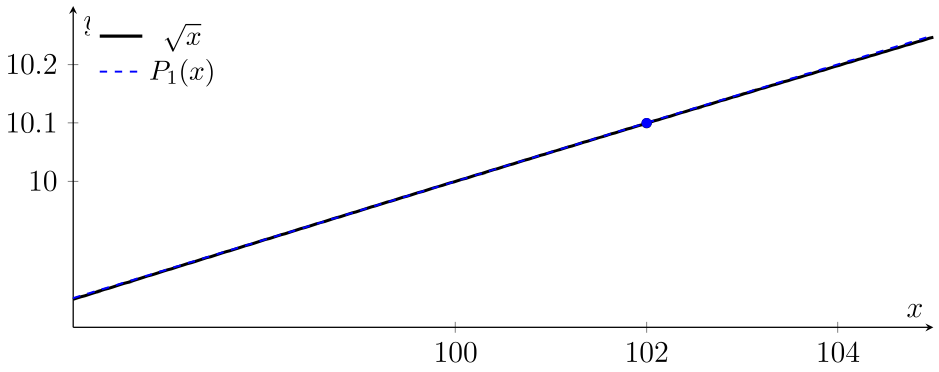


Figure 40: Choosing $a = 100$ keeps the target close to the centre.

Example 18.8 (Approximating $\ln(1.2)$)**.** For $f(x) = \ln x$ centred at $a = 1$, the polynomial through order four is

$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

Substitution gives

$$P_4(1.2) = 0.182266\overline{6}.$$

The next derivative is

$$f^{(5)}(x) = \frac{24}{x^5}.$$

On $[1, 1.2]$, its absolute value is at most 24. Hence,

$$|R_4(1.2)| \leqslant \frac{24}{5!}(0.2)^5 = 6.4 \times 10^{-5}.$$

Therefore,

$$\ln(1.2) = 0.182266\overline{6} + R_4, \qquad |R_4| \leqslant 6.4 \times 10^{-5}.$$

18.5 Choosing a sufficient order

Example 18.9 (Approximating $e^{0.5}$ within 10^{-6})**.** For every derivative of e^x, we may use the simple bound

$$e^t < 2 \qquad \text{on } [0, 0.5].$$

Thus the Maclaurin remainder after order n satisfies

$$|R_n(0.5)| \leqslant \frac{2}{(n+1)!}\left(\frac{1}{2}\right)^{n+1}.$$

For $n = 6$,

$$\frac{2}{7!\,2^7} = \frac{1}{322560} > 10^{-6}.$$

This estimate does not provide the desired guarantee. For $n = 7$,

$$\frac{2}{8!\,2^8} = \frac{1}{5160960} < 10^{-6}.$$

Therefore, order seven is sufficient. The approximation is

$$\begin{aligned} P_7(0.5) &= 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} + \cdots + \frac{0.5^7}{7!} \\ &= 1.648721168\ldots \end{aligned}$$

with a guaranteed error smaller than 10^{-6}.

Common mistakes in error estimates include the following.

- Bounding $f^{(n)}$ instead of the next derivative $f^{(n+1)}$.
- Using only $\left|f^{(n+1)}(a)\right|$ instead of a bound on the whole interval.

- Forgetting the factorial $(n+1)!$.
- Replacing $|x-a|^{n+1}$ by $|x|^{n+1}$ when $a \neq 0$.
- Reporting a numerical approximation without its error guarantee.
- Assuming that every infinitely differentiable function equals an infinite Taylor series. Our results concern finite Taylor polynomials and finite remainders.