

Week 3: Derivatives

Analysis of Proposed Solutions — Answer Key

Instructions. Each proposed solution below **might contain a mathematical error**. In addition, every proposed solution contains one incorrect English word or expression. The color of a possible-solution box does not indicate correctness. The right solution and the correction of the English mistake appear immediately afterward in a black box.

Exercise

(1). Let

$$f(x) = \begin{cases} x^3 \sin(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Decide whether f is differentiable at 0.

Possible solution

For $x \neq 0$, the product and chain rules give

$$f'(x) = 3x^2 \sin(1/x^2) - 2 \cos(1/x^2).$$

The term $\cos(1/x^2)$ has no limit as $x \rightarrow 0$. Therefore, $f'(0)$ does not exist.

Right solution

The proposed mathematics is incorrect. The derivative at 0 must be calculated from the definition:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} h^2 \sin(1/h^2) = 0,$$

because $|h^2 \sin(1/h^2)| \leq h^2$. Thus, f is differentiable at 0 and $f'(0) = 0$. The formula for $f'(x)$ when $x \neq 0$ does not determine $f'(0)$.

The English correction is: “does not *exist*,” not “does not exists.”

(2). Determine whether $g(x) = |x^3|$ is differentiable at 0.

Possible solution

Since $g(0) = 0$, we consider the difference quotient. If $h > 0$, then

$$\frac{g(h) - g(0)}{h} = h^2,$$

whereas, if $h < 0$, then

$$\frac{g(h) - g(0)}{h} = -h^2.$$

The two one-sided limits are equals to 0. Therefore, g is differentiable at 0 and $g'(0) = 0$.

Right solution

The proposed mathematics is correct. Both one-sided difference quotients tend to 0, so

$$g'(0) = 0.$$

The English correction is: “The two one-sided limits are *equal* to 0,” not “are equals to 0.”

(3). For which values of a is

$$h(x) = \begin{cases} ax + 1, & x \leq 2, \\ x^2 - 1, & x > 2 \end{cases}$$

differentiable at $x = 2$?

Possible solution

The derivative in the left is a , and the derivative on the right is $2x$. Matching the derivatives at $x = 2$ gives

$$a = 4,$$

so h is differentiable at 2 precisely when $a = 4$.

Right solution

The proposed mathematics is incorrect because continuity must be checked first. Continuity at 2 requires

$$2a + 1 = 2^2 - 1 = 3,$$

so $a = 1$. For this value, the left derivative is 1, whereas the right derivative is 4. Therefore, there is **no value of a** for which h is differentiable at 2.

The English correction is: “the derivative *from* the left,” not “the derivative in the left.”

(4). Find the tangent line to the curve

$$x^3 + y^2 = 1$$

at $(1, 0)$.

Possible solution

Implicit differentiation gives

$$3x^2 + 2yy' = 0, \quad y' = -\frac{3x^2}{2y}.$$

At $(1, 0)$ the formula is undefined. Hence, the curve has no tangent line in $(1, 0)$.

Right solution

The proposed mathematics is incorrect. An undefined value of y' may indicate a vertical tangent. Writing the tangent relation in differential form gives

$$3x^2 dx + 2y dy = 0.$$

At $(1, 0)$ this becomes $3 dx = 0$, so the tangent direction is vertical. Therefore, the tangent line is

$$x = 1.$$

The English correction is: “at $(1, 0)$,” not “in $(1, 0)$.”

(5). Let $p : [0, +\infty) \rightarrow [2, +\infty)$ be $p(x) = x^3 + 2$. Is p^{-1} differentiable at 2?

Possible solution

Since $p^{-1}(2) = 0$, the inverse-function formula gives

$$(p^{-1})'(2) = \frac{1}{p'(0)} = \frac{1}{0} = +\infty.$$

Thus, p^{-1} is differentiable in 2 and its derivative is $+\infty$.

Right solution

The proposed mathematics is incorrect. The inverse-function formula cannot be used because $p'(0) = 0$. In fact,

$$p^{-1}(x) = \sqrt[3]{x-2}, \quad x \geq 2.$$

Its right difference quotient at 2 is

$$\frac{p^{-1}(2+h) - p^{-1}(2)}{h} = \frac{h^{1/3}}{h} = h^{-2/3} \rightarrow +\infty.$$

Since a derivative must be finite, p^{-1} is not differentiable at 2.

The English correction is: “differentiable *at* 2,” not “differentiable in 2.”

(6). The power dissipated by a resistor is

$$P(t) = \frac{V(t)^2}{R(t)}.$$

At one instant, $V = 12$, $V' = 0.5$, $R = 6$ and $R' = 0.2$. Find P' .

Possible solution

Both V and R changes with time. The quotient and chain rules give

$$P' = \frac{2VV'}{R} - \frac{V^2R'}{R^2}.$$

At the indicated instant,

$$P' = \frac{2(12)(0.5)}{6} - \frac{(12)^2(0.2)}{6^2} = 1.2.$$

Thus, the dissipated power is increasing at 1.2 power units per unit of time.

Right solution

The proposed mathematics is correct. Indeed,

$$P' = \frac{2VV'}{R} - \frac{V^2R'}{R^2} = 2 - 0.8 = 1.2.$$

The English correction is: “Both V and R *change* with time,” not “changes with time.”

(7). Differentiate

$$q(x) = (3 + \cos x)^{e^x}.$$

Possible solution

Taking logarithms give

$$\ln q = e^x \ln(3 + \cos x).$$

Using the product rule,

$$\frac{q'}{q} = e^x \ln(3 + \cos x) - \frac{e^x \sin x}{3 + \cos x}.$$

Therefore,

$$q'(x) = (3 + \cos x)^{e^x} e^x \left(\ln(3 + \cos x) - \frac{\sin x}{3 + \cos x} \right).$$

Right solution

The proposed mathematics is correct. Logarithmic differentiation gives

$$q'(x) = (3 + \cos x)^{e^x} e^x \left(\ln(3 + \cos x) - \frac{\sin x}{3 + \cos x} \right).$$

The English correction is: “Taking logarithms *gives*,” not “give.”

(8). Current I and voltage V satisfy

$$I^2 + VI = 8.$$

Find dI/dV at the operating point $(V, I) = (2, 2)$.

Possible solution

Differentiating with respect of V gives

$$2I \frac{dI}{dV} + V \frac{dI}{dV} = 0.$$

At $(2, 2)$, we obtain $6 dI/dV = 0$, so $dI/dV = 0$.

Right solution

The proposed mathematics is incorrect because differentiating VI requires the product rule:

$$2I \frac{dI}{dV} + I + V \frac{dI}{dV} = 0.$$

Consequently,

$$\frac{dI}{dV} = -\frac{I}{2I + V}.$$

At $(V, I) = (2, 2)$,

$$\frac{dI}{dV} = -\frac{2}{6} = -\frac{1}{3}.$$

The English correction is: “with respect to V ,” not “with respect of V .”

(9). True or false?

Suppose that f is differentiable at 0, $f(0) = 0$ and $f'(0) = 0$. Then the function

$$g(x) = |f(x)|$$

is differentiable at 0 and $g'(0) = 0$.

Possible solution

The statement is false. The absolute value creates a corner at every zeros of f , so g cannot be differentiable at 0.

Right solution

The statement is **true**. Since $f(0) = 0$, we have $g(0) = 0$. Moreover, for $h \neq 0$,

$$\left| \frac{g(h) - g(0)}{h} \right| = \frac{|f(h)|}{|h|} = \left| \frac{f(h) - f(0)}{h} \right| \rightarrow |f'(0)| = 0.$$

Therefore,

$$\lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = 0,$$

so g is differentiable at 0 and $g'(0) = 0$.

The English correction is: “at every *zero* of f ,” not “at every zeros of f .”