
Choose the Next Move

Week 4: Mean Value Theorems and indeterminate limits

Five problems. Fifteen decisions. One card at a time.

How the game works

1. Discuss each decision with your group.
2. When time is called, every group shows one card: **A**, **B**, **C** or **D**.
3. All groups show their cards simultaneously. Speed gives no advantage.
4. After the answer is revealed, one group that chose correctly explains it.
5. A different student represents the group in each problem.

2 points Correct card

+1 point Clear mathematical justification

The three highest-scoring groups receive 0.5 points in Midterm 1.

Problem 1 – Step 1: may we use Rolle's theorem?

SHOW YOUR CARD

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

Which argument correctly establishes that such a point must exist?

A f is continuous on $[-1, 1]$, differentiable on $(-1, 1)$ and $f(-1) = f(1)$.

B f is odd, so necessarily $f'(0) = 0$.

C f has three zeros, so Rolle's theorem immediately identifies every zero of f' .

D Continuity on $[-1, 1]$ alone guarantees a point where $f' = 0$.

Problem 1 – Step 1: answer

CORRECT ANSWER

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

A. All three hypotheses of Rolle's theorem hold.

The function is a polynomial, so it is continuous on the closed interval and differentiable on the open interval.

Moreover,

$$f(-1) = 0 = f(1).$$

Therefore, Rolle's theorem guarantees at least one $c \in (-1, 1)$ for which $f'(c) = 0$. Oddness alone would not imply $f'(0) = 0$.

Problem 1 – Step 2: what does the theorem guarantee?

SHOW YOUR CARD

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

Before calculating, what follows from Rolle's theorem alone?

A There is exactly one $c \in (-1, 1)$ such that $f'(c) = 0$.

B There is at least one $c \in (-1, 1)$ such that $f'(c) = 0$.

C Every zero of f in $[-1, 1]$ is also a zero of f' .

D At least one endpoint must satisfy $f'(c) = 0$.

Problem 1 – Step 2: answer

CORRECT ANSWER

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

B. At least one $c \in (-1, 1)$ satisfies $f'(c) = 0$.

Rolle's theorem is an existence theorem. It neither identifies the point nor guarantees uniqueness. The point belongs to the open interval, not necessarily to an endpoint.

Problem 1 – Step 3: find every possible value

SHOW YOUR CARD

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

Since $f'(x) = 3x^2 - 1$, which is the complete set of solutions in $(-1, 1)$?

A $\{0\}$

B $\{-1, 1\}$

C $\left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$

D $\left\{\frac{1}{\sqrt{3}}\right\}$

Problem 1 – Step 3: answer

CORRECT ANSWER

Problem 1

Let

$$f(x) = x^3 - x$$

on $[-1, 1]$.

Determine whether there exists $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

If so, find every such value of c .

$$\text{C. } c = \pm \frac{1}{\sqrt{3}}.$$

We solve

$$3c^2 - 1 = 0 \quad \Longleftrightarrow \quad c^2 = \frac{1}{3}.$$

Both solutions lie in $(-1, 1)$. The theorem guarantees at least one point; the direct calculation shows that there are two.

Problem 2 – Step 1: determine the monotonicity

SHOW YOUR CARD

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

Since

$$F'(x) = 15x^2(x - 1)(x + 1),$$

which monotonicity description is correct?

- A** F increases outside $[-1, 1]$ and decreases on $(-1, 1)$; the zero of F' at 0 causes no sign change.
- B** F changes monotonicity at each of -1 , 0 and 1 .
- C** $F'(x) \geq 0$ everywhere because $x^2 \geq 0$, so F is increasing on $\mathbb{R}$.
- D** F decreases outside $[-1, 1]$ and increases inside it.

Problem 2 – Step 1: answer

CORRECT ANSWER

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

A. The sign changes at -1 and 1 , but not at 0 .

The factor x^2 is non-negative and has even multiplicity.
Hence

x	$(-\infty, -1)$	$(-1, 1) \setminus \{0\}$	$(1, +\infty)$
$F'(x)$	+	-	+

Although $F'(0) = 0$, the derivative is negative on both sides of 0 . Thus 0 is a stationary point, but not a local extremum.

Problem 2 – Step 2: locate the zeros

SHOW YOUR CARD

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

Also $F(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $F(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. Where are the zeros located?

A Exactly one in each of $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.

B Exactly one in each of $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, +\infty)$.

C Exactly one in $(-1, 1)$ and none outside this interval.

D Exactly one in each outer interval and none in $(-1, 1)$.

Problem 2 – Step 2: answer

CORRECT ANSWER

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

A. There is exactly one zero in each of the three monotonicity intervals.

On $(-\infty, -1)$, F increases from $-\infty$ to 1. On $(-1, 1)$, it decreases from 1 to -3 . On $(1, +\infty)$, it increases from -3 to $+\infty$.

Continuity gives a zero in each interval, while strict monotonicity guarantees that each one is unique.

Problem 2 – Step 3: state the exact conclusion

SHOW YOUR CARD

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

Which argument gives the exact number of real solutions?

- A** The Intermediate Value Theorem and strict monotonicity give one zero in each of three intervals: three in total.
- B** Three sign changes and degree five imply exactly five real zeros.
- C** Three zeros of F' imply four zeros of F by the converse of Rolle's theorem.
- D** The point $x = 0$ is stationary, so it must be an additional solution.

Problem 2 – Step 3: answer

CORRECT ANSWER

Problem 2

Let

$$F(x) = 3x^5 - 5x^3 - 1.$$

Determine the exact number of real solutions of

$$3x^5 - 5x^3 = 1.$$

Useful values are $F(-1) = 1$,
 $F(0) = -1$ and $F(1) = -3$.

A. The equation has exactly three real solutions.

The interval analysis gives three distinct zeros and proves that no additional zero is possible. Therefore,

$$3x^5 - 5x^3 = 1 \text{ has exactly three real solutions.}$$

The degree gives only an upper bound of five real zeros, and Rolle's theorem does not have a converse.

Problem 3 – Step 1: translate the inequality

SHOW YOUR CARD

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

Which choice gives the correct function and the correct target inequality?

A $G(x) = x - 1 - \ln x$; prove that $G(x) \geq 0$.

B $G(x) = \ln x - x + 1$; prove that $G(x) \geq 0$.

C $G(x) = x - 1 - \ln x$; prove that $G'(x) \geq 0$ for every $x > 0$.

D $G(x) = x \ln x - 1$; prove that its only zero is 1.

Problem 3 – Step 1: answer

CORRECT ANSWER

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

A. Set $G(x) = x - 1 - \ln x$ and prove $G(x) \geq 0$.

The desired inequality is equivalent to

$$x - 1 - \ln x \geq 0.$$

Notice that the domain is $(0, +\infty)$. We now locate the minimum of G on this interval.

Problem 3 – Step 2: locate the minimum

SHOW YOUR CARD

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

Since $G'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$, which conclusion is valid?

- A** G is increasing on its whole domain because $x > 0$.
- B** G decreases on $(0, 1)$ and increases on $(1, +\infty)$.
- C** G increases on $(0, 1)$ and decreases on $(1, +\infty)$.
- D** $x = 1$ is a maximum because $G'(1) = 0$.

Problem 3 – Step 2: answer

CORRECT ANSWER

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

B. G decreases before 1 and increases after 1.

Because $x > 0$, the sign of

$$G'(x) = \frac{x - 1}{x}$$

is determined by $x - 1$. Thus $G' < 0$ on $(0, 1)$, $G'(1) = 0$, and $G' > 0$ on $(1, +\infty)$. Therefore, G attains its global minimum at $x = 1$.

Problem 3 – Step 3: conclude precisely

SHOW YOUR CARD

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

What follows from the monotonicity analysis?

- A** $G(x) > 0$ for every $x > 0$, so equality never occurs.
- B** $G(x) \geq 0$, and equality holds exactly at $x = 1$.
- C** $G(x) \leq 0$, and equality holds exactly at $x = 1$.
- D** $G(1) = 0$, but this gives no information away from $x = 1$.

Problem 3 – Step 3: answer

CORRECT ANSWER

Problem 3

Prove that

$$\ln x \leq x - 1 \quad (x > 0),$$

and determine precisely when equality holds.

B. $G(x) \geq 0$, with equality exactly when $x = 1$.

At the global minimum,

$$G(1) = 1 - 1 - \ln 1 = 0.$$

Hence

$$\boxed{\ln x \leq x - 1 \quad (x > 0)},$$

and equality holds only for $x = 1$.

Problem 4 – Step 1: is L'Hospital's rule legal?

SHOW YOUR CARD

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

Which is the correct first conclusion?

- A** The quotient is undefined at 0, so L'Hospital's rule cannot be used.
- B** The numerator tends to 0, so the limit is 0.
- C** Direct substitution produces 0/0, so L'Hospital's rule may be applied under its hypotheses.
- D** We must differentiate the entire quotient using the quotient rule.

Problem 4 – Step 1: answer

CORRECT ANSWER

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

C. The limit has the indeterminate form $0/0$.

We have

$$\sin 0 - 0 = 0, \quad 0^3 = 0.$$

The fact that the quotient is not defined at the limiting point is not an obstacle: limits concern nearby points. L'Hospital's rule applies on a punctured neighbourhood of 0.

Problem 4 – Step 2: differentiate correctly

SHOW YOUR CARD

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

What quotient results from the first application of L'Hospital's rule?

A $\frac{\cos x}{3x^2}$

B $\frac{\cos x - 1}{3x^2}$

C $\frac{\cos x - x}{3x^2}$

D $\frac{\cos x - 1}{x^2}$

Problem 4 – Step 2: answer

CORRECT ANSWER

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

$$\text{B. } \frac{\cos x - 1}{3x^2}.$$

Differentiate numerator and denominator separately:

$$(\sin x - x)' = \cos x - 1, \quad (x^3)' = 3x^2.$$

The new quotient is still of the form $0/0$, so another application is permitted.

Problem 4 – Step 3: complete the calculation

SHOW YOUR CARD

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

Which continuation is completely correct?

A $\frac{\cos x - 1}{3x^2} \longrightarrow \frac{-\sin x}{6x} \longrightarrow \frac{-\cos x}{6}$, so the limit is $-1/6$.

B After the second application the numerator tends to 0, so the limit is 0.

C $\frac{\cos x - 1}{3x^2} \longrightarrow \frac{\sin x}{6x}$, so the limit is $1/6$.

D Each application preserves the form $0/0$, so the limit does not exist.

Problem 4 – Step 3: answer

CORRECT ANSWER

Problem 4

Calculate

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}.$$

At every stage, verify that the resulting quotient has an indeterminate form before applying L'Hospital's rule again.

A. Three justified applications give $-1/6$.

After each of the first two applications, the new quotient is again $0/0$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{-\sin x}{6x} \\ &= \lim_{x \rightarrow 0} \frac{-\cos x}{6} = -\frac{1}{6}.\end{aligned}$$

Problem 5 – Step 1: identify the hidden form

SHOW YOUR CARD

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

What is the correct classification and first transformation?

- A** It is a $0/0$ form; differentiate the base and exponent separately.
- B** It is a 1^∞ form; introduce y and take logarithms.
- C** It is a 0^0 form; direct substitution gives 1.
- D** It is an ∞/∞ form; apply L'Hospital's rule directly.

Problem 5 – Step 1: answer

CORRECT ANSWER

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

B. It is a 1^∞ form, so we first take logarithms.

As $x \rightarrow 0^+$,

$$\sin x + \cos x \rightarrow 1, \quad \frac{1}{x} \rightarrow +\infty.$$

The base is positive for sufficiently small positive x .
The original expression is a power, not a quotient, so L'Hospital's rule does not yet apply.

Problem 5 – Step 2: reveal the quotient

SHOW YOUR CARD

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

Set $y = (\sin x + \cos x)^{1/x}$. Which identity is correct?

A $\ln y = x \ln(\sin x + \cos x)$

B $\ln y = \frac{1}{x \ln(\sin x + \cos x)}$

C $\ln y = \frac{\ln(\sin x + \cos x)}{x}$

D $\ln y = \frac{\sin x + \cos x}{x}$

Problem 5 – Step 2: answer

CORRECT ANSWER

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

$$\text{C. } \ln y = \frac{\ln(\sin x + \cos x)}{x}.$$

The exponent becomes a factor after taking logarithms. The resulting quotient has the form

$$\frac{0}{0},$$

because $\ln(\sin x + \cos x) \rightarrow \ln 1 = 0$. L'Hospital's rule can now be applied.

Problem 5 – Step 3: return from the logarithm

SHOW YOUR CARD

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

Which conclusion is correct?

- A** The logarithmic limit is 1, so the original limit is also 1.
- B** The logarithmic limit is e , so the original limit is e .
- C** The logarithmic limit is 1, so the original limit is e .
- D** The base tends to 1, so the exponent is irrelevant.

Problem 5 – Step 3: answer

CORRECT ANSWER

Problem 5

Calculate

$$\lim_{x \rightarrow 0^+} (\sin x + \cos x)^{1/x}.$$

Explain why L'Hospital's rule cannot be applied directly to the original expression.

C. The logarithmic limit is 1, and the original limit is e .

L'Hospital's rule gives

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\cos x - \sin x}{\sin x + \cos x} = 1.$$

Therefore,

$$y = e^{\ln y} \longrightarrow e^1 = e,$$

and the required limit is $\boxed{e}$.

Tie-breaker

SHOW YOUR CARD

True or false?

Let f be differentiable on $\mathbb{R}$. If

$$f'(x) > 0$$

for every $x \in \mathbb{R}$, then the equation $f(x) = 0$ has exactly one real solution.

Each tied group must submit both:

- its true-or-false decision;
- a complete proof or a counterexample.

No points are awarded for an unsupported answer.

Tie-breaker: answer

CORRECT ANSWER

True or false?

Let f be differentiable on $\mathbb{R}$. If $f'(x) > 0$ for every $x \in \mathbb{R}$, then $f(x) = 0$ has exactly one real solution.

False. The hypothesis gives uniqueness, not existence.

The function is strictly increasing, so it has at most one zero. It need not have any zero. For example,

$$f(x) = e^x$$

satisfies $f'(x) = e^x > 0$ for every $x \in \mathbb{R}$, but $f(x) > 0$ everywhere.