Week 8

Lecture 13

Introduction to the Riemann Integral

Sheldon Dantas
FEL CTU Prague

2026–2027 · 1st semester

What we will build today

By the end of this lecture, you should be able to

- approximate a curved region by rectangles;
- construct lower, upper and Riemann sums;
- state what it means for a bounded function to be Riemann integrable;
- calculate simple integrals directly from sums;
- interpret a definite integral as signed area;
- use the main algebraic and order properties of the integral.

From area to partitions

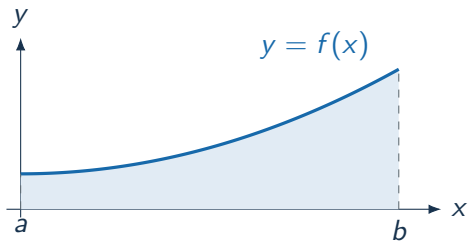
Lecture 13 · Week 8

Why do we need a new limiting process?

The area of a rectangle is easy:

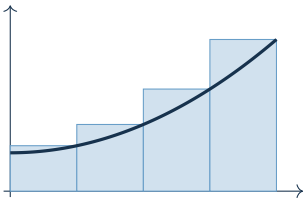
$$\text{area} = \text{base} \times \text{height}.$$

For a curved boundary, the height changes continuously. We therefore replace the region by many thin rectangles.

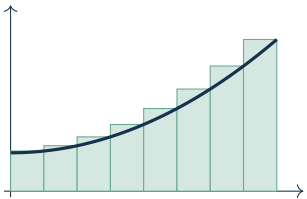


The integral will be the limit of these rectangle approximations.

More rectangles give a better approximation



4 rectangles



8 rectangles

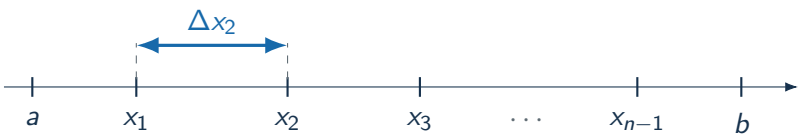
A partition divides the interval

A **partition** of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b.$$

The ith subinterval and its width are

$$I_i = [x_{i-1}, x_i], \qquad \Delta x_i = x_i - x_{i-1}.$$



Lower and upper sums

Lecture 13 · Week 8

Each subinterval gives two natural heights

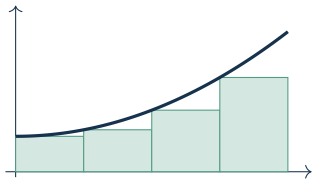
Let $f : [a, b] \to \mathbb{R}$ be bounded. On $I_i = [x_{i-1}, x_i]$, define

$$m_i = \inf_{x \in I_i} f(x), \qquad M_i = \sup_{x \in I_i} f(x).$$

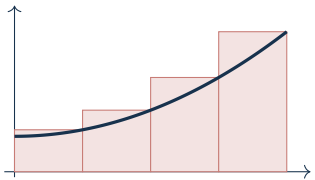
The corresponding sums are

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Lower and upper rectangles trap the area



lower sum



upper sum

For every partition, $L(f, P) \leqslant U(f, P)$.

Refinement squeezes the two sums together

A partition Q is a **refinement** of P if it contains every point of P and possibly some additional points.

If Q refines P, then

$$L(f, P) \leqslant L(f, Q) \leqslant U(f, Q) \leqslant U(f, P).$$



The function is integrable when this gap can be made arbitrarily small.

Definition of the Riemann integral

Let $f : [a, b] \to \mathbb{R}$ be bounded. Define

$$\underline{\int_a^b} f = \sup_P L(f, P), \qquad \overline{\int_a^b} f = \inf_P U(f, P).$$

We say that f is **Riemann integrable** on $[a, b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Their common value is denoted by

$$\int_a^b f(x)\, dx.$$

The practical integrability criterion

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if, for every $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i,$$

the gap becomes small when the function oscillates very little on sufficiently short subintervals.

Two integrals calculated from sums

Lecture 13 · Week 8

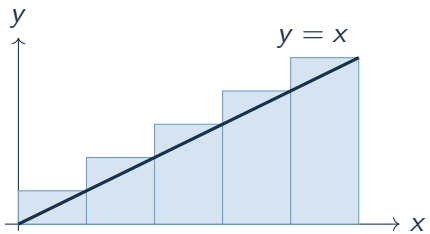
Example 1: calculate $\int_0^1 x\,dx$

Use the equal partition

$$x_i = \frac{i}{n}, \qquad \Delta x = \frac{1}{n}.$$

Because $f(x) = x$ is increasing,

$$m_i = f(x_{i-1}) = \frac{i-1}{n}, \qquad M_i = f(x_i) = \frac{i}{n}.$$



Step 1: compute the lower sum

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(i-1) \\ &= \frac{1}{n^2}\frac{n(n-1)}{2} \\ &= \frac{n-1}{2n}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} L(f, P_n) = \frac{1}{2}.$$

Step 2: compute the upper sum

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} \frac{i}{n} \frac{1}{n} \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i \\ &= \frac{1}{n^2} \frac{n(n+1)}{2} \\ &= \frac{n+1}{2n}. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} U(f, P_n) = \frac{1}{2}.$$

Step 3: squeeze the integral

For every n,

$$\frac{n-1}{2n} = L(f, P_n) \leqslant \underline{\int_0^1} x\,dx \leqslant \overline{\int_0^1} x\,dx \leqslant U(f, P_n) = \frac{n+1}{2n}.$$

Both outer quantities tend to $1/2$. Hence, the lower and upper integrals coincide.

$$\int_0^1 x\,dx = \frac{1}{2}.$$

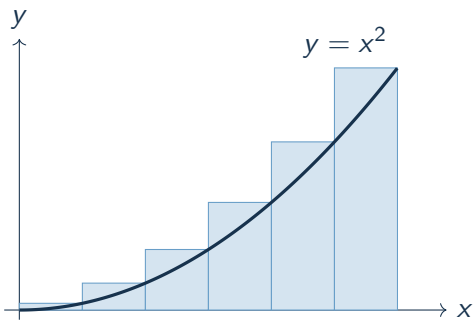
This agrees with the elementary area of a triangle:

$$\frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$

Example 2: calculate $\int_0^1 x^2\,dx$

Take n equal subintervals and use the right endpoints

$$x_i = \frac{i}{n}, \qquad \Delta x = \frac{1}{n}.$$



Compute the Riemann sum

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

Letting $n \to \infty$ gives

$$\frac{(1+1/n)(2+1/n)}{6} \longrightarrow \frac{1}{3}.$$

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}.}$$

Integrability and essential properties

Lecture 13 · Week 8

Which functions are Riemann integrable?

The following sufficient conditions cover most functions used in Calculus 1.

Continuous functions

Every function continuous on a closed bounded interval $[a, b]$ is Riemann integrable there.

Monotone functions

Every bounded monotone function on $[a, b]$ is Riemann integrable there.

A bounded function with only finitely many discontinuities is also Riemann integrable.

Changing finitely many values does not change the integral

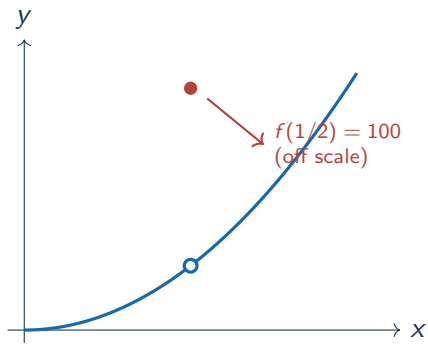
Consider

$$f(x) = \begin{cases} x^2, & x \neq \frac{1}{2}, \\ 100, & x = \frac{1}{2}. \end{cases}$$

The exceptional point can affect at most one rectangle. As the mesh tends to zero, the width of that rectangle tends to zero.

Therefore,

$$\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$



Linearity and additivity

If f and g are integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

If $c \in [a, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

We may split an interval wherever the formula or the sign of the function changes.

Order and absolute-value estimates

If $f(x) \leqslant g(x)$ for every $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

In particular, if $m \leqslant f(x) \leqslant M$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

Also,

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$$

The integral respects inequalities.

Orientation and zero-length intervals

For every integrable function,

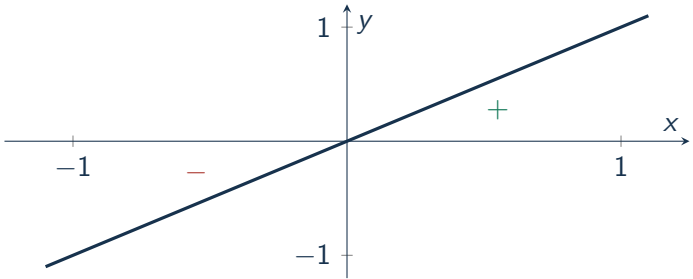
$$\int_a^a f(x)\,dx = 0.$$

Reversing the limits changes the sign:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

A definite integral is signed area

For $f(x) = x$ on $[-1, 1]$, the areas on the two sides of the origin are equal, but their signs are opposite.



Thus,

$$\int_{-1}^{1} x \, dx = 0, \qquad \int_{-1}^{1} |x| \, dx = 1.$$

Symmetry can eliminate half the work

On a symmetric interval $[-a, a]$:

Odd function

If $f(-x) = -f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

The signed areas cancel.

Even function

If $f(-x) = f(x)$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

The two halves agree.

Examples: x^3 is odd, while x^2 and $\cos x$ are even.

Quick conceptual check

Decide whether each statement is true or false.

1. If $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for every $x \in [a, b]$.
2. A function must be continuous in order to be Riemann integrable.
3. Changing the value of an integrable function at one point changes its integral.

Answers to the conceptual check

1. **False.** Positive and negative signed areas may cancel, as for $f(x) = x$ on $[-1, 1]$.
2. **False.** Bounded monotone functions and bounded functions with finitely many discontinuities are integrable.
3. **False.** Changing finitely many values does not change the integral.

Next lecture

Today we answered:

What is a definite integral?

In the next lecture, we answer:

How can we calculate it efficiently?