Calculus 1

Taylor Polynomial and the Lagrange Remainder

Two solved exercises for group work

Problem 1

Let

$$f(x) = e^x \cos x.$$

(a) Calculate the derivatives needed to construct the fourth-order Taylor polynomial P_4 of f centred at $a = 0$.

(b) Use the Lagrange remainder to prove that

$$|f(x) - P_4(x)| \leqslant \frac{\sqrt{2e}}{960} \qquad \text{for every } x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Solution.

We differentiate one step at a time:

$$\begin{aligned} f(x) &= e^x \cos x, \\ f'(x) &= e^x(\cos x - \sin x), \\ f''(x) &= -2e^x \sin x, \\ f^{(3)}(x) &= -2e^x(\sin x + \cos x), \\ f^{(4)}(x) &= -4e^x \cos x, \\ f^{(5)}(x) &= 4e^x(\sin x - \cos x). \end{aligned}$$

At $a = 0$,

$$f(0) = 1, \qquad f'(0) = 1, \qquad f''(0) = 0, \qquad f^{(3)}(0) = -2, \qquad f^{(4)}(0) = -4.$$

Therefore,

$$\boxed{P_4(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}.}$$

For every $x \in [-1/2, 1/2]$, the Lagrange remainder has the form

$$f(x) - P_4(x) = \frac{f^{(5)}(c_x)}{5!} x^5,$$

where c_x lies between 0 and x. On $[-1/2, 1/2]$, we have $e^x \leqslant e^{1/2}$. Moreover,

$$\sin x - \cos x = \sqrt{2} \sin\left(x - \frac{\pi}{4}\right),$$

and hence $|\sin x - \cos x| \leqslant \sqrt{2}$. Thus,

$$\left|f^{(5)}(x)\right| = 4e^x \, |\sin x - \cos x| \leqslant 4e^{1/2}\sqrt{2} = 4\sqrt{2e}.$$

Since $|x| \leqslant 1/2$,

$$|f(x) - P_4(x)| \leqslant \frac{4\sqrt{2e}}{5!} \left(\frac{1}{2}\right)^5 = \boxed{\frac{\sqrt{2e}}{960}}.$$

Problem 2

We want to decide which of the two numbers

$$e^{\pi} \qquad \text{and} \qquad \pi^{e}$$

is larger. Define

$$F(x) = \frac{x}{e} - \ln x, \qquad x > 0.$$

(a) Calculate the derivatives needed to construct the fourth-order Taylor polynomial $P_4(x)$ of F centred at $a = e$. Write the polynomial explicitly as a function of x before substituting $x = \pi$.

(b) Evaluate $P_4(\pi)$ and use it to approximate

$$G = \ln\left(\frac{e^{\pi}}{\pi^{e}}\right) = \pi - e\ln\pi.$$

(c) Use the Lagrange remainder to prove that the error in the approximation of G is smaller than 5×10^{-5}.

(d) Use the resulting certified approximation to decide which of e^{π} and π^{e} is larger.

Hint. Observe that

$$G = eF(\pi).$$

Solution.

We calculate the derivatives:

$$\begin{aligned}
F(x) &= \frac{x}{e} - \ln x, \\
F'(x) &= \frac{1}{e} - \frac{1}{x}, \\
F''(x) &= \frac{1}{x^2}, \\
F^{(3)}(x) &= -\frac{2}{x^3}, \\
F^{(4)}(x) &= \frac{6}{x^4}, \\
F^{(5)}(x) &= -\frac{24}{x^5}.
\end{aligned}$$

At $a = e$,

$$F(e) = 0, \qquad F'(e) = 0, \qquad F''(e) = \frac{1}{e^2}, \qquad F^{(3)}(e) = -\frac{2}{e^3}, \qquad F^{(4)}(e) = \frac{6}{e^4}.$$

Therefore, the exact fourth-order Taylor polynomial is

$$\boxed{P_4(x) = \frac{(x-e)^2}{2e^2} - \frac{(x-e)^3}{3e^3} + \frac{(x-e)^4}{4e^4}.}$$

Substituting $x = \pi$ into the polynomial gives

$$P_4(\pi) = \frac{(\pi-e)^2}{2e^2} - \frac{(\pi-e)^3}{3e^3} + \frac{(\pi-e)^4}{4e^4}.$$

Since $G = eF(\pi)$, we approximate G by

$$G_4 = eP_4(\pi) = \frac{(\pi-e)^2}{2e} - \frac{(\pi-e)^3}{3e^2} + \frac{(\pi-e)^4}{4e^3} \approx 0.0299383.$$

The Lagrange remainder gives a point $c \in (e, \pi)$ such that

$$F(\pi) = P_4(\pi) + R_4(\pi), \qquad R_4(\pi) = \frac{F^{(5)}(c)}{5!}(\pi - e)^5.$$

The function $24/x^5$ is decreasing on $[e, \pi]$, so

$$\left|F^{(5)}(x)\right| \leqslant \frac{24}{e^5} \qquad \text{for every } x \in [e, \pi].$$

Consequently,

$$|G - G_4| = e\,|R_4(\pi)| \leqslant \frac{(\pi - e)^5}{5e^4} \approx 4.9791 \times 10^{-5} < 5 \times 10^{-5}.$$

Thus, the true value of G belongs to the certified interval

$$0.0299383 - 0.00005 < G < 0.0299383 + 0.00005,$$

that is,

$$\boxed{0.0298883 < G < 0.0299883.}$$

In particular, $G > 0$. Since

$$G = \ln\left(\frac{e^\pi}{\pi^e}\right) > 0,$$

we have $e^\pi/\pi^e > 1$. Therefore,

$$\boxed{e^\pi > \pi^e.}$$