Calculus 1

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Integration: Exercises

1 The Fundamental Theorem of Calculus

Exercise 1. Find the derivative of each of the following functions:

(a) $F(x) = \displaystyle\int_0^{x^2} \sin\big(\ln(1+t)\big)\,\mathrm{d}t;$

(b) $F(x) = \displaystyle\int_{x^2}^{1} \sin^3(t)\,\mathrm{d}t;$

(c) $F(x) = \displaystyle\int_{x^2}^{x^3} \cos^3(t)\,\mathrm{d}t.$

Exercise 2. Study the intervals on which the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x^3-x^2} e^{-t^2}\,\mathrm{d}t$$

is increasing or decreasing. Then determine its local extrema.

Exercise 3. Calculate the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{x\int_0^x \sin(t^2)\,\mathrm{d}t}{\sin(x^4)};$

(b) $\displaystyle\lim_{x\to 0} \frac{\int_{x^2+x}^{\sin x} e^{-t^2}\,\mathrm{d}t}{\sin^2 x};$

(c) $\displaystyle\lim_{x\to+\infty} \frac{\int_0^{x^2} \ln(\sqrt{t})\,\mathrm{d}t}{2x^2\ln x - x^2};$

(d) $\displaystyle\lim_{x\to 0} \frac{\int_0^{\sqrt{x}} \arcsin(t^2)\,\mathrm{d}t}{\sqrt{x}\sin x}.$

Exercise 4. Calculate the following limit:

$$\lim_{x\to+\infty} \frac{\sqrt{\ln x}}{x} \int_2^x \frac{\mathrm{d}y}{\sqrt{\ln(y^2)}}.$$

Exercise 5. Find the absolute maximum of the function $f : [1, +\infty) \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t} \right) \, \mathrm{d}t.$$

Given that

$$\lim_{x \to +\infty} f(x) = \frac{1}{2} \left(\sqrt{\pi} - 1 \right),$$

find the absolute minimum of f.

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 e^{-x}$.

(a) Find an antiderivative F of f.

(b) Determine, if they exist, the points at which the slope of the tangent line to the graph of F is smallest and the points at which it is largest.

Exercise 7. Prove that, for every $x \in \mathbb{R}^+$,

$$\int_1^{\cosh x} \sqrt{t^2 - 1} \, \mathrm{d}t = \frac{\cosh x \, \sinh x}{2} - \frac{x}{2}.$$

Exercise 8. Consider the function

$$f(x) = \int_0^{x^3 - x^2} e^{-t^2} \, \mathrm{d}t, \qquad x \in \mathbb{R}.$$

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the local extrema of f.

(c) Calculate

$$\lim_{x \to 0} \frac{f(x)}{\sin(x^3 - x^2)}.$$

Exercise 9. Let $f : (0, +\infty) \to \mathbb{R}$ be defined by

$$f(x) = \int_0^{1/x} e^{-t^2} \, \mathrm{d}t.$$

(a) Study the intervals of monotonicity of f.

(b) Given that

$$\int_0^{+\infty} e^{-t^2} \, \mathrm{d}t = \frac{\sqrt{\pi}}{2},$$

determine the range $f\big((0, +\infty)\big)$.

Exercise 10. Let $f : (0, \pi/2) \to \mathbb{R}$ be defined by

$$f(x) = \int_{1/e}^{\tan x} \frac{t}{1 + t^2} \, \mathrm{d}t + \int_{1/e}^{\cot x} \frac{1}{t(1 + t^2)} \, \mathrm{d}t.$$

(a) Study the differentiability of f and calculate $f'(x)$.

(b) Calculate $f(\pi/4)$.

(c) What conclusion can you draw from the previous parts?

Exercise 11. Find the Taylor polynomial of order 2, centred at the origin, for the function

$$f(x) = \int_x^{x^2} \cos(t^2)\,\mathrm{d}t.$$

2 Antiderivatives

2.1 Basic antiderivatives and substitution

Exercise 12. Find the following antiderivatives:

(a) $\displaystyle\int 2x^5\,\mathrm{d}x;$

(b) $\displaystyle\int \frac{\mathrm{d}x}{2x^5};$

(c) $\displaystyle\int (4x^7+2)^5 x^6\,\mathrm{d}x;$

(d) $\displaystyle\int \frac{2x+1}{1+x^2}\,\mathrm{d}x;$

(e) $\displaystyle\int \left(\sqrt[3]{1+x}\right)^4\,\mathrm{d}x;$

(f) $\displaystyle\int \sin(x/2)\,\mathrm{d}x;$

(g) $\displaystyle\int \tan(2x)\,\mathrm{d}x;$

(h) $\displaystyle\int \frac{\cos x}{\sin^4 x}\,\mathrm{d}x;$

(i) $\displaystyle\int \cos(e^x)e^x\,\mathrm{d}x;$

(j) $\displaystyle\int \frac{\sqrt[3]{1+\ln x}}{x}\,\mathrm{d}x;$

(k) $\displaystyle\int \frac{\mathrm{d}x}{e^x+1};$

(l) $\displaystyle\int x(2x+5)^{10}\,\mathrm{d}x.$

2.2 Integration by parts

Exercise 13. Find the following antiderivatives:

(a) $\int \ln x \, \mathrm{d}x$;

(b) $\int \arctan x \, \mathrm{d}x$;

(c) $\int \arcsin x \, \mathrm{d}x$;

(d) $\int x \sin x \, \mathrm{d}x$;

(e) $\int x e^{-x} \, \mathrm{d}x$;

(f) $\int x^2 e^{3x} \, \mathrm{d}x$;

(g) $\int x \sin x \cos x \, \mathrm{d}x$;

(h) $\int e^{2x} \cos(5x) \, \mathrm{d}x$;

(i) $\int x \ln x \, \mathrm{d}x$.

2.3 Integration of rational functions

Exercise 14. Find the following antiderivatives:

(a) $\int \dfrac{x+1}{x^2-2x+1} \, \mathrm{d}x$;

(b) $\int \dfrac{x^3}{x^3+2x^2-x-2} \, \mathrm{d}x$;

(c) $\int \dfrac{x}{(x+1)(x-1)^2} \, \mathrm{d}x$;

(d) $\int \dfrac{\mathrm{d}x}{3x^2+4}$;

(e) $\int \dfrac{2x}{x^2+3} \, \mathrm{d}x$;

(f) $\int \dfrac{\mathrm{d}x}{(x^2-1)(x^2+1)}$;

(g) $\int \dfrac{\mathrm{d}x}{(x^2+4)^2}$;

(h) $\displaystyle\int \frac{(x-1)x^2(2x+3)}{(x^2+1)^2}\,\mathrm{d}x;$

(i) $\displaystyle\int \frac{2x^3-3x+7}{x+1}\,\mathrm{d}x.$

2.4 Integration of trigonometric functions

Exercise 15. Find the following antiderivatives:

(a) $\displaystyle\int \sin^3 x\,\mathrm{d}x;$

(b) $\displaystyle\int \cos^4 x\,\mathrm{d}x;$

(c) $\displaystyle\int \tan^5 x\,\mathrm{d}x;$

(d) $\displaystyle\int \cos^3(5x)\,\mathrm{d}x;$

(e) $\displaystyle\int \tan^2 x \sec^2 x\,\mathrm{d}x;$

(f) $\displaystyle\int \sin^2 x \cos^2 x\,\mathrm{d}x;$

(g) $\displaystyle\int \sin^2 x \cos^3 x\,\mathrm{d}x;$

(h) $\displaystyle\int \sin^3 x \cos^3 x\,\mathrm{d}x;$

(i) $\displaystyle\int \frac{\mathrm{d}x}{\sin x};$

(j) $\displaystyle\int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x;$

(k) $\displaystyle\int \frac{\mathrm{d}x}{\cos^3 x};$

(l) $\displaystyle\int \frac{\sin^5 x}{\cos^3 x}\,\mathrm{d}x.$

Exercise 16. Find the following antiderivatives and definite integral:

(a) $\displaystyle\int \frac{\cos x}{1+\cos x}\,\mathrm{d}x;$

(b) $\displaystyle\int \frac{1+\tan x}{1-\tan x}\,\mathrm{d}x;$

(c) $\displaystyle\int \frac{\mathrm{d}x}{1+\cos^2(3x)}$;

(d) $\displaystyle\int \frac{\mathrm{d}x}{3\sin^2 x + 5\cos^2 x}$;

(e) $\displaystyle\int \frac{\sin(2x)}{1+\sin^2 x}\,\mathrm{d}x$;

(f) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos x}\,\mathrm{d}x$.

3 Improper integrals

Exercise 17. Prove that the following integrals exist and have the indicated values:

(a) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{1+e^x} = 1 + \ln\left(\frac{2}{1+e}\right)$;

(b) $\displaystyle\int_1^{+\infty} \frac{x-1}{x^3-3x^2+x+5}\,\mathrm{d}x = \frac{3\pi+\ln 2}{10}$;

(c) $\displaystyle\int_0^{+\infty} \frac{x}{3+x^4}\,\mathrm{d}x = \frac{\sqrt{3}\,\pi}{12}$;

(d) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{e^x+e^{-x}} = \frac{\pi}{2}$;

(e) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{2}$;

(f) $\displaystyle\int_{-\pi}^{\pi} (1+\cos x)^2\,\mathrm{d}x = 3\pi$;

(g) $\displaystyle\int_{-\pi/2}^{\pi/2} |\sin x|^3\,\mathrm{d}x = \frac{4}{3}$;

(h) $\displaystyle\int_0^{\pi/2} \sin^2 y \cos^2 y\,\mathrm{d}y = \frac{\pi}{16}$.