Calculus 1

Integration: Exercises with Solutions

Problem 1. Calculate

$$\int \left(2e^{x/2} - \frac{1}{1-\sqrt{x}}\right) dx.$$

Solution. The expression is defined for $x \geqslant 0$, with $x \neq 1$. We integrate the first term directly:

$$\int 2e^{x/2}\, dx = 4e^{x/2}.$$

For the second term, put $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\, du$. Hence,

$$\begin{aligned} -\int \frac{dx}{1-\sqrt{x}} &= -2\int \frac{u}{1-u}\, du = \int \left(2 - \frac{2}{1-u}\right) du \\ &= 2u + 2\ln|1-u| + C. \end{aligned}$$

Therefore, on every interval contained in the domain,

$$\boxed{4e^{x/2} + 2\sqrt{x} + 2\ln\left|1-\sqrt{x}\right| + C.}$$

Problem 2. Calculate

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}, \qquad x > 0.$$

Solution. The exponents $1/2$ and $1/3$ suggest the substitution $x = u^6$. Then $dx = 6u^5\, du$, and

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = 6\int \frac{u^5}{u^3 + u^2}\, du = 6\int \frac{u^3}{u+1}\, du.$$

Polynomial division gives

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Consequently,

$$6\int \frac{u^3}{u+1}\, du = 2u^3 - 3u^2 + 6u - 6\ln(u+1) + C.$$

Since $u = x^{1/6}$, the answer is

$$\boxed{2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(\sqrt[6]{x} + 1\right) + C.}$$

Problem 3. Calculate

$$\int \frac{e^{2x}}{e^{2x}+1}\, dx.$$

Solution. Put $u = e^{2x} + 1$. Then $du = 2e^{2x}\,dx$, so

$$\int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \boxed{\frac{1}{2}\ln\bigl(e^{2x}+1\bigr) + C.}$$

Problem 4. Calculate

$$\int \frac{e^{x}}{e^{2x}+1}\,dx.$$

Solution. With $u = e^x$, we have $du = e^x\,dx$. Therefore,

$$\int \frac{e^{x}}{e^{2x}+1}\,dx = \int \frac{du}{u^2+1} = \boxed{\arctan(e^x) + C.}$$

Problem 5. Calculate

$$\int x^2 \sin\left(\frac{x}{2}\right)\,dx.$$

Solution. We apply integration by parts with

$$u = x^2, \qquad dv = \sin(x/2)\,dx, \qquad du = 2x\,dx, \qquad v = -2\cos(x/2).$$

Thus,

$$\int x^2 \sin(x/2)\,dx = -2x^2\cos(x/2) + 4\int x\cos(x/2)\,dx.$$

For the remaining integral, integrate by parts once more:

$$\int x\cos(x/2)\,dx = 2x\sin(x/2) + 4\cos(x/2) + C.$$

Therefore,

$$\boxed{-2x^2\cos(x/2) + 8x\sin(x/2) + 16\cos(x/2) + C.}$$

Problem 6. Calculate

$$\int x\ln(2x)\,dx, \qquad x > 0.$$

Solution. Take $u = \ln(2x)$ and $dv = x\,dx$. Then $du = dx/x$ and $v = x^2/2$. Hence,

$$\begin{aligned}\int x\ln(2x)\,dx &= \frac{x^2}{2}\ln(2x) - \frac{1}{2}\int x\,dx \\ &= \boxed{\frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + C.}\end{aligned}$$

Problem 7. Calculate

$$\int \arctan x\,dx.$$

Solution. Write the integrand as $1 \cdot \arctan x$ and use integration by parts:

$$u = \arctan x, \qquad dv = dx, \qquad du = \frac{dx}{1+x^2}, \qquad v = x.$$

Then

$$\begin{aligned}\int \arctan x \, dx &= x \arctan x - \int \frac{x}{1+x^2}\, dx \\ &= \boxed{x \arctan x - \frac{1}{2}\ln(1+x^2) + C.}\end{aligned}$$

Problem 8. Calculate

$$\int x^3 \sin(x^2)\, dx.$$

Solution. Let $u = x^2$. Since $du = 2x\, dx$, we have $x^3\, dx = u\, du/2$. Thus,

$$\int x^3 \sin(x^2)\, dx = \frac{1}{2}\int u \sin u \, du.$$

Integration by parts gives

$$\int u \sin u \, du = -u\cos u + \sin u + C.$$

Consequently,

$$\boxed{-\frac{x^2}{2}\cos(x^2) + \frac{1}{2}\sin(x^2) + C.}$$

Problem 9. Calculate

$$\int \sin(\sqrt{x})\, dx, \qquad x \geqslant 0.$$

Solution. Put $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\, du$. Hence,

$$\int \sin(\sqrt{x})\, dx = 2\int u \sin u \, du.$$

Using integration by parts,

$$2\int u \sin u \, du = -2u\cos u + 2\sin u + C.$$

Therefore,

$$\boxed{-2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C.}$$

Problem 10. Calculate

$$\int \arcsin x \, dx, \qquad -1 < x < 1.$$

Solution. Take $u = \arcsin x$ and $dv = dx$. Then

$$du = \frac{dx}{\sqrt{1-x^2}}, \qquad v = x.$$

Thus,

$$\begin{aligned}\int \arcsin x\, dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= \boxed{x \arcsin x + \sqrt{1-x^2} + C.}\end{aligned}$$

Problem 11. Calculate

$$\int \frac{\arctan(\sqrt{x})}{\sqrt{x}}\, dx, \qquad x > 0.$$

Solution. Put $u = \sqrt{x}$, so $dx = 2u\, du$. The integral becomes

$$2\int \arctan u\, du.$$

Using the result of Problem 7,

$$2\int \arctan u\, du = 2u \arctan u - \ln(1+u^2) + C.$$

Hence,

$$\boxed{2\sqrt{x} \arctan(\sqrt{x}) - \ln(1+x) + C.}$$

Problem 12. Calculate

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\, dx, \qquad x > 0.$$

Solution. Let $x = u^6$. Then $dx = 6u^5\, du$, and

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\, dx = 6\int \frac{u^8}{1+u^2}\, du.$$

Polynomial division gives

$$\frac{u^8}{1+u^2} = u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}.$$

Therefore,

$$6\int \frac{u^8}{1+u^2}\, du = \frac{6}{7}u^7 - \frac{6}{5}u^5 + 2u^3 - 6u + 6\arctan u + C.$$

Returning to x,

$$\boxed{\frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\arctan\left(\sqrt[6]{x}\right) + C.}$$

Problem 13. Decompose the rational function into elementary terms and calculate its integral:

$$\int \frac{x^3 - 7x - 24}{x^2 - 2x - 8}\, dx.$$

Solution. Since the numerator has larger degree, we first divide the polynomials:

$$\frac{x^3 - 7x - 24}{x^2 - 2x - 8} = x + 2 + \frac{5x - 8}{(x-4)(x+2)}.$$

For the proper fraction, write

$$\frac{5x - 8}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}.$$

Substitution of $x = 4$ and $x = -2$ gives $A = 2$ and $B = 3$. Hence,

$$\frac{x^3 - 7x - 24}{x^2 - 2x - 8} = x + 2 + \frac{2}{x-4} + \frac{3}{x+2}.$$

Therefore,

$$\boxed{\frac{x^2}{2} + 2x + 2\ln|x-4| + 3\ln|x+2| + C.}$$

Problem 14. Decompose into partial fractions:

$$\frac{-2x + 5}{(x-1)^2(x+2)}.$$

Solution. The repeated factor $(x-1)^2$ requires two terms:

$$\frac{-2x + 5}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$

After multiplying by $(x-1)^2(x+2)$,

$$-2x + 5 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

Putting $x = 1$ gives $B = 1$, and putting $x = -2$ gives $C = 1$. Taking $x = 0$ then gives $A = -1$. Thus,

$$\boxed{-\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+2}.}$$

Problem 15. Calculate

$$\int \frac{x^2 + 2x + 1}{x(x^2+1)}\, dx.$$

Solution. We seek constants A, B, C such that

$$\frac{x^2 + 2x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1}.$$

Comparing coefficients in

$$x^2 + 2x + 1 = A(x^2 + 1) + (Bx + C)x$$

gives $A = 1$, $B = 0$ and $C = 2$. Hence,

$$\int \left(\frac{1}{x} + \frac{2}{x^2 + 1} \right) dx = \boxed{\ln|x| + 2\arctan x + C.}$$

Problem 16. Calculate

$$\int \frac{2x^2 + 3x + 18}{x^3 + 9x}\, dx.$$

Solution. Since $x^3 + 9x = x(x^2 + 9)$, write

$$\frac{2x^2 + 3x + 18}{x(x^2 + 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}.$$

Comparing coefficients yields $A = 2$, $B = 0$ and $C = 3$. Therefore,

$$\boxed{2\ln|x| + \arctan\left(\frac{x}{3}\right) + C.}$$

Problem 17. Calculate

$$\int \frac{5x^3 + 6x + 2}{x^2(x^2 + 2x + 2)}\, dx.$$

Solution. The appropriate decomposition is

$$\frac{5x^3 + 6x + 2}{x^2(x^2 + 2x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 2x + 2}.$$

After clearing denominators and comparing coefficients, we obtain

$$A = 2, \qquad B = 1, \qquad C = 3, \qquad D = -5.$$

Thus,

$$\frac{5x^3 + 6x + 2}{x^2(x^2 + 2x + 2)} = \frac{2}{x} + \frac{1}{x^2} + \frac{3x - 5}{x^2 + 2x + 2}.$$

Since $3x - 5 = \frac{3}{2}(2x + 2) - 8$ and $x^2 + 2x + 2 = (x + 1)^2 + 1$, integration gives

$$\boxed{2\ln|x| - \frac{1}{x} + \frac{3}{2}\ln(x^2 + 2x + 2) - 8\arctan(x + 1) + C.}$$

Problem 18. Calculate

$$\int 18x^2 \sqrt[4]{6x^3+5}\,dx.$$

Solution. Put $u = 6x^3 + 5$. Then $du = 18x^2\,dx$, and hence

$$\int 18x^2(6x^3+5)^{1/4}\,dx = \int u^{1/4}\,du = \boxed{\frac{4}{5}(6x^3+5)^{5/4} + C.}$$

Problem 19. Calculate

$$\int \frac{2t^3+1}{(t^4+2t)^3}\,dt.$$

Solution. Let $u = t^4 + 2t$. Since $du = (4t^3+2)dt = 2(2t^3+1)dt$, we obtain

$$\int \frac{2t^3+1}{(t^4+2t)^3}\,dt = \frac{1}{2}\int u^{-3}\,du = \boxed{-\frac{1}{4(t^4+2t)^2} + C.}$$

Problem 20. Calculate

$$\int \frac{1+e^x}{1-e^{-x}}\,dx.$$

Solution. First multiply the numerator and denominator by e^x:

$$\frac{1+e^x}{1-e^{-x}} = \frac{e^x(1+e^x)}{e^x-1}.$$

Put $u = e^x$, so $du = e^x\,dx$. Then

$$\int \frac{e^x(1+e^x)}{e^x-1}\,dx = \int \frac{1+u}{u-1}\,du = \int \left(1 + \frac{2}{u-1}\right) du.$$

Therefore, on every interval not containing 0,

$$\boxed{e^x + 2\ln|e^x-1| + C.}$$

Problem 21. Calculate

$$\int \ln(1+x^2)\,dx.$$

Solution. Use integration by parts with $u = \ln(1+x^2)$ and $dv = dx$. Then

$$\begin{aligned}\int \ln(1+x^2)\,dx &= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx \\ &= x\ln(1+x^2) - 2\int \left(1 - \frac{1}{1+x^2}\right) dx.\end{aligned}$$

Hence,

$$\boxed{x\ln(1+x^2) - 2x + 2\arctan x + C.}$$

Problem 22. Calculate

$$\int \frac{3t+5}{(t^2+1)^2}\,dt.$$

Solution. Separate the numerator:

$$\int \frac{3t}{(t^2+1)^2}\,dt + 5\int \frac{dt}{(t^2+1)^2}.$$

The first integral is

$$-\frac{3}{2(t^2+1)}.$$

For the second, the identity

$$\int \frac{dt}{(t^2+1)^2} = \frac{t}{2(t^2+1)} + \frac{1}{2}\arctan t + C$$

may be verified by differentiation. Combining the terms gives

$$\boxed{\frac{5t-3}{2(t^2+1)} + \frac{5}{2}\arctan t + C.}$$

Problem 23. Calculate

$$\int \frac{4x+1}{x^2-2x+5}\,dx.$$

Solution. Since

$$4x + 1 = 2(2x-2) + 5,$$

we have

$$\begin{aligned}\int \frac{4x+1}{x^2-2x+5}\,dx &= 2\int \frac{2x-2}{x^2-2x+5}\,dx + 5\int \frac{dx}{(x-1)^2+4} \\ &= \boxed{2\ln(x^2-2x+5) + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.}\end{aligned}$$

Problem 24. Calculate

$$\int \sqrt{\frac{1-x}{x}}\,dx, \qquad 0 < x < 1.$$

Solution. Put $x = \sin^2\theta$, where $0 < \theta < \pi/2$. Then

$$dx = 2\sin\theta\cos\theta\,d\theta, \qquad \sqrt{\frac{1-x}{x}} = \frac{\cos\theta}{\sin\theta}.$$

Therefore,

$$\int \sqrt{\frac{1-x}{x}}\,dx = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C.$$

Returning to x,

$$\boxed{\arcsin(\sqrt{x}) + \sqrt{x(1-x)} + C.}$$

Problem 25. Calculate

$$\int e^x \sin x\,dx.$$

Solution. Let $I = \int e^x \sin x\,dx$. Integrating by parts twice gives

$$I = e^x \sin x - \int e^x \cos x\,dx,$$
$$\int e^x \cos x\,dx = e^x \cos x + I.$$

Substituting the second identity into the first,

$$2I = e^x(\sin x - \cos x).$$

Thus,

$$\boxed{\frac{e^x}{2}(\sin x - \cos x) + C.}$$

Problem 26. Calculate

$$\int (x+1)e^{3x-1}\,dx.$$

Solution. Use integration by parts with $u = x+1$ and $dv = e^{3x-1}dx$. Then $du = dx$ and $v = e^{3x-1}/3$. Hence,

$$\int (x+1)e^{3x-1}\,dx = \frac{x+1}{3}e^{3x-1} - \frac{1}{3}\int e^{3x-1}\,dx$$
$$= \boxed{\frac{3x+2}{9}e^{3x-1} + C.}$$

Problem 27. Calculate

$$\int (x+\sqrt{x})\ln x\,dx, \qquad x > 0.$$

Solution. We integrate the two terms separately. Integration by parts gives

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4},$$

and

$$\int x^{1/2} \ln x \, dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2}.$$

Therefore,

$$\boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} + \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C.}$$

Problem 28. Calculate

$$\int (\ln x)^2 \, dx, \qquad x > 0.$$

Solution. Use integration by parts with $u = (\ln x)^2$ and $dv = dx$:

$$\begin{aligned} \int (\ln x)^2 \, dx &= x(\ln x)^2 - 2 \int \ln x \, dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + C. \end{aligned}$$

Hence,

$$\boxed{x(\ln x)^2 - 2x \ln x + 2x + C.}$$

Problem 29. Calculate

$$\int x(\ln x)^3 \, dx, \qquad x > 0.$$

Solution. First take $u = (\ln x)^3$ and $dv = x \, dx$:

$$\int x(\ln x)^3 dx = \frac{x^2}{2} (\ln x)^3 - \frac{3}{2} \int x(\ln x)^2 dx.$$

Applying integration by parts twice more to the remaining powers of $\ln x$ gives

$$\boxed{\frac{x^2}{2} \left((\ln x)^3 - \frac{3}{2} (\ln x)^2 + \frac{3}{2} \ln x - \frac{3}{4} \right) + C.}$$

Problem 30. Calculate

$$\int \frac{(4 \sin x + 6) \cos x}{(\sin x - 1)(\sin^2 x + 4)} \, dx.$$

Solution. Put $u = \sin x$, so $du = \cos x \, dx$. Then

$$\frac{4u + 6}{(u - 1)(u^2 + 4)} = \frac{2}{u - 1} + \frac{-2u + 2}{u^2 + 4}.$$

Therefore,

$$\int \frac{4u + 6}{(u - 1)(u^2 + 4)} \, du = 2 \ln |u - 1| - \ln(u^2 + 4) + \arctan(u/2) + C.$$

Returning to x,

$$\boxed{2\ln|\sin x - 1| - \ln(\sin^2 x + 4) + \arctan\left(\frac{\sin x}{2}\right) + C.}$$

Problem 31. Calculate

$$\int \frac{x \sin x}{\cos^3 x}\, dx.$$

Solution. Use integration by parts with

$$u = x, \qquad dv = \frac{\sin x}{\cos^3 x}\, dx.$$

Since

$$v = \frac{1}{2\cos^2 x},$$

we obtain

$$\begin{aligned}\int \frac{x \sin x}{\cos^3 x}\, dx &= \frac{x}{2\cos^2 x} - \frac{1}{2}\int \frac{dx}{\cos^2 x} \\ &= \boxed{\frac{x}{2\cos^2 x} - \frac{1}{2}\tan x + C.}\end{aligned}$$

Problem 32. Calculate

$$\int \frac{6}{e^{2x} - e^x - 2}\, dx.$$

Solution. Put $u = e^x$, so $dx = du/u$. Since $u^2 - u - 2 = (u-2)(u+1)$,

$$\int \frac{6}{e^{2x} - e^x - 2}\, dx = \int \frac{6}{u(u-2)(u+1)}\, du.$$

The partial-fraction decomposition is

$$\frac{6}{u(u-2)(u+1)} = -\frac{3}{u} + \frac{1}{u-2} + \frac{2}{u+1}.$$

Consequently,

$$\boxed{-3x + \ln|e^x - 2| + 2\ln(e^x + 1) + C,}$$

on every interval not containing $\ln 2$.

Problem 33. Suppose that f is integrable on every interval required below and that

$$\int_{12}^{-10} f(x)\,dx = 6, \qquad \int_{100}^{-10} f(x)\,dx = -2, \qquad \int_{100}^{-5} f(x)\,dx = 4.$$

Calculate $\displaystyle\int_{-5}^{12} f(x)\,dx$.

Solution. We follow a path from -5 to 12 through 100 and -10:

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx.$$

Reversing the limits in the first and third given integrals gives

$$\int_{-5}^{100} f(x)\,dx = -4, \qquad \int_{-10}^{12} f(x)\,dx = -6.$$

Therefore,

$$\boxed{\int_{-5}^{12} f(x)\,dx = -4 - 2 - 6 = -12.}$$

Problem 34. Differentiate the following functions:

$$\textbf{(a)} \quad g(x) = \int_{-4}^{x} e^{2t}\cos^2(1-5t)\,dt,$$

$$\textbf{(b)} \quad h(x) = \int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt.$$

Solution. **(a)** By the Fundamental Theorem of Calculus,

$$\boxed{g'(x) = e^{2x}\cos^2(1-5x).}$$

(b) The variable limit is the lower limit. The chain rule therefore gives

$$\begin{aligned} h'(x) &= -\frac{(x^2)^4+1}{(x^2)^2+1}\cdot 2x \\ &= \boxed{-\frac{2x(x^8+1)}{x^4+1}.} \end{aligned}$$

Problem 35. Calculate

$$\int_{1}^{4} (2x^3 - 5x)\,dx.$$

Solution. An antiderivative is

$$F(x) = \frac{x^4}{2} - \frac{5x^2}{2}.$$

Thus,

$$\int_1^4 (2x^3 - 5x)\,dx = \left[\frac{x^4}{2} - \frac{5x^2}{2}\right]_1^4$$
$$= (128 - 40) - \left(\frac{1}{2} - \frac{5}{2}\right) = \boxed{90.}$$

Problem 36. Calculate

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx.$$

Solution. Put $u = \sin x$, so $du = \cos x\,dx$. The limits become $u = 0$ and $u = 1$. Therefore,

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \int_0^1 \frac{du}{1 + u^2} = [\arctan u]_0^1 = \boxed{\frac{\pi}{4}.}$$

Problem 37. Calculate

$$\int_0^1 \arctan x\,dx.$$

Solution. From integration by parts,

$$\int \arctan x\,dx = x \arctan x - \frac{1}{2}\ln(1 + x^2) + C.$$

Hence,

$$\int_0^1 \arctan x\,dx = \left[x \arctan x - \frac{1}{2}\ln(1 + x^2)\right]_0^1$$
$$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2.}$$

Problem 38. Calculate

$$\int_0^{\pi/4} \frac{\tan^7 x}{\cos^2 x}\,dx.$$

Solution. Put $u = \tan x$. Then $du = dx/\cos^2 x$, and the limits are 0 and 1. Thus,

$$\int_0^{\pi/4} \frac{\tan^7 x}{\cos^2 x}\,dx = \int_0^1 u^7\,du = \left[\frac{u^8}{8}\right]_0^1 = \boxed{\frac{1}{8}.}$$

Problem 39. Calculate

$$\int_1^e \frac{\ln x}{x^2}\,dx.$$

Solution. Use integration by parts with $u = \ln x$ and $dv = x^{-2}dx$. Then $du = dx/x$ and $v = -1/x$. Therefore,

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

It follows that

$$\int_1^e \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e = \boxed{1 - \frac{2}{e}.}$$

Problem 40. Calculate

$$\int_1^2 \frac{2x^3 + 6x^2 - 1}{x^2(x+1)^2}\,dx.$$

Solution. Partial fractions give

$$\frac{2x^3 + 6x^2 - 1}{x^2(x+1)^2} = \frac{2}{x} - \frac{1}{x^2} + \frac{3}{(x+1)^2}.$$

Hence,

$$\int \frac{2x^3 + 6x^2 - 1}{x^2(x+1)^2}\,dx = 2\ln x + \frac{1}{x} - \frac{3}{x+1} + C$$

on $[1, 2]$. Evaluating at the endpoints,

$$\begin{aligned}\int_1^2 \frac{2x^3 + 6x^2 - 1}{x^2(x+1)^2}\,dx &= \left[2\ln x + \frac{1}{x} - \frac{3}{x+1}\right]_1^2 \\ &= \boxed{2\ln 2.}\end{aligned}$$

Problem 41. Calculate

$$\int_0^2 (x-1)\,|x-1|\,e^x\,dx.$$

Solution. The absolute value changes formula at $x = 1$:

$$(x-1)\,|x-1| = \begin{cases} -(x-1)^2, & 0 \leqslant x < 1, \\ (x-1)^2, & 1 \leqslant x \leqslant 2. \end{cases}$$

An antiderivative of $(x-1)^2 e^x$ is

$$F(x) = e^x(x^2 - 4x + 5).$$

Therefore,

$$\begin{aligned}\int_0^2 (x-1)\,|x-1|\,e^x\,dx &= -\left[F(x)\right]_0^1 + \left[F(x)\right]_1^2 \\ &= -(2e-5) + (e^2 - 2e) \\ &= \boxed{e^2 - 4e + 5.}\end{aligned}$$

Problem 42. Let $f(x) = x^2 + 3x + 2$ on $[1, 4]$. Find every number $c \in (1, 4)$ supplied by the Mean Value Theorem for integrals; that is, find c such that

$$\int_1^4 f(x)\,dx = f(c)(4 - 1).$$

Solution. First calculate the integral:

$$\int_1^4 (x^2 + 3x + 2)\,dx = \left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_1^4 = \frac{99}{2}.$$

Thus, $f(c)$ must equal the average value

$$\frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{33}{2}.$$

We solve

$$c^2 + 3c + 2 = \frac{33}{2},$$

or equivalently $2c^2 + 6c - 29 = 0$. Hence,

$$c = \frac{-3 \pm \sqrt{67}}{2}.$$

Only one of these values belongs to $(1, 4)$, so

$$\boxed{c = \frac{-3 + \sqrt{67}}{2}.}$$

Problem 43. Determine whether the improper integral converges and, if it does, calculate it:

$$\int_0^\infty \frac{-1}{(x+2)(x+1)}\,dx.$$

Solution. We have

$$\frac{-1}{(x+2)(x+1)} = \frac{1}{x+2} - \frac{1}{x+1}.$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{-1}{(x+2)(x+1)}\,dx &= \lim_{R\to\infty} \left[\ln(x+2) - \ln(x+1)\right]_0^R \\ &= \lim_{R\to\infty} \ln\left(\frac{R+2}{R+1}\right) - \ln 2 \\ &= \boxed{-\ln 2.}\end{aligned}$$

Thus, the integral converges.

Problem 44. Determine whether the improper integral converges and, if it does, calculate it:

$$\int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

Solution. An integral over the whole real line must be split at a finite point, for example at 0:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{A\to-\infty}\int_A^0 \frac{dx}{1+x^2} + \lim_{B\to\infty}\int_0^B \frac{dx}{1+x^2} \\ &= \lim_{A\to-\infty}[\arctan x]_A^0 + \lim_{B\to\infty}[\arctan x]_0^B \\ &= \frac{\pi}{2}+\frac{\pi}{2} = \boxed{\pi.}\end{aligned}$$

Both component integrals converge, so the original integral converges.

Problem 45. Calculate

$$\int_1^{\infty} e^{-2x-3}\,dx.$$

Solution. By definition,

$$\begin{aligned}\int_1^{\infty} e^{-2x-3}\,dx &= \lim_{R\to\infty}\left[-\frac{1}{2}e^{-2x-3}\right]_1^R \\ &= \lim_{R\to\infty}\left(-\frac{1}{2}e^{-2R-3}+\frac{1}{2}e^{-5}\right) \\ &= \boxed{\frac{1}{2}e^{-5}.}\end{aligned}$$

Problem 46. Calculate

$$\int_0^{\infty} \frac{dx}{e^x+1}.$$

Solution. Let $u = e^x$. Then $dx = du/u$, and the limits become 1 and ∞:

$$\int_0^{\infty} \frac{dx}{e^x+1} = \int_1^{\infty} \frac{du}{u(u+1)}.$$

Since

$$\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1},$$

we obtain

$$\begin{aligned}\int_1^{\infty} \frac{du}{u(u+1)} &= \lim_{R\to\infty}[\ln u - \ln(u+1)]_1^R \\ &= 0-(-\ln 2) = \boxed{\ln 2.}\end{aligned}$$

Problem 47. Determine whether the improper integral converges:

$$\int_0^{\pi/2} \frac{dx}{\sin x}.$$

Solution. The integrand is unbounded at $x = 0$. Since

$$\int \frac{dx}{\sin x} = \ln|\tan(x/2)| + C,$$

we have

$$\begin{aligned}\int_0^{\pi/2} \frac{dx}{\sin x} &= \lim_{\varepsilon\to 0^+} \big[\ln\big(\tan(x/2)\big)\big]_\varepsilon^{\pi/2} \\ &= -\lim_{\varepsilon\to 0^+} \ln\big(\tan(\varepsilon/2)\big) = +\infty.\end{aligned}$$

Therefore, the integral **diverges**.

Problem 48. Calculate

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx.$$

Solution. Put $u = e^x$, so $du = e^x\,dx$. The limits become 1 and ∞. Hence,

$$\begin{aligned}\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx &= \int_1^\infty \frac{du}{u^2+1} \\ &= \lim_{R\to\infty} [\arctan u]_1^R = \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}.}\end{aligned}$$

Problem 49. Calculate

$$\int_{-\infty}^{-1} \frac{dx}{x\sqrt{1-x}}.$$

Solution. Put $u = \sqrt{1-x}$. Then $x = 1 - u^2$ and $dx = -2u\,du$. When $x \to -\infty$, $u \to \infty$, and when $x = -1$, $u = \sqrt{2}$. Thus,

$$\begin{aligned}\int_{-\infty}^{-1} \frac{dx}{x\sqrt{1-x}} &= \int_\infty^{\sqrt{2}} \frac{-2}{1-u^2}\,du = \int_\infty^{\sqrt{2}} \frac{2}{u^2-1}\,du \\ &= \left[\ln\left(\frac{u-1}{u+1}\right)\right]_\infty^{\sqrt{2}} \\ &= \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \boxed{-2\ln(\sqrt{2}+1).}\end{aligned}$$

Problem 50. Calculate

$$\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx.$$

Solution. Put $u = 1/x$. Then $du = -dx/x^2$. As $x \to 0^+$, $u \to \infty$, and as $x \to \infty$, $u \to 0^+$. Therefore,

$$\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx = \int_\infty^0 -e^{-u}\,du = \int_0^\infty e^{-u}\,du = \boxed{1.}$$

Problem 51. Calculate

$$\int_0^\infty \frac{dx}{x^2-2x+5}.$$

Solution. Complete the square:

$$x^2-2x+5=(x-1)^2+4.$$

Thus,

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2-2x+5} &= \lim_{R\to\infty}\left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_0^R \\ &= \boxed{\frac{\pi}{4}+\frac{1}{2}\arctan\left(\frac{1}{2}\right).}\end{aligned}$$

Problem 52. Determine whether the improper integral converges:

$$\int_{-1}^1 \frac{dx}{x^2-2\left|x\right|+1}.$$

Solution. For $-1<x<0$, the denominator equals

$$x^2+2x+1=(x+1)^2,$$

while for $0<x<1$ it equals

$$x^2-2x+1=(x-1)^2.$$

Therefore,

$$\int_{-1}^1 \frac{dx}{x^2-2\left|x\right|+1}=\int_{-1}^0 \frac{dx}{(x+1)^2}+\int_0^1 \frac{dx}{(x-1)^2}.$$

Each integral diverges to $+\infty$ at one endpoint. Hence, the original integral **diverges**.

Problem 53. Calculate the following improper integrals:

$$\textbf{(a)}\quad \int_0^\infty xe^{-2x}\,dx, \qquad\qquad \textbf{(b)}\quad \int_0^\infty e^{-x}\sin x\,dx.$$

Solution. **(a)** Integration by parts gives

$$\int xe^{-2x}\,dx=-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C.$$

Since $xe^{-2x}\to 0$ as $x\to\infty$,

$$\int_0^\infty xe^{-2x}\,dx=\left[-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}\right]_0^\infty=\boxed{\frac{1}{4}.}$$

(b) Let $I=\int e^{-x}\sin x\,dx$. Two integrations by parts give

$$I=-\frac{1}{2}e^{-x}(\sin x+\cos x)+C.$$

Therefore,

$$\int_0^\infty e^{-x}\sin x\,dx=\left[-\frac{1}{2}e^{-x}(\sin x+\cos x)\right]_0^\infty=\boxed{\frac{1}{2}.}$$