20 Lecture #13: Week 8

20.1 From areas to sums

The area of a rectangle is easy to calculate:

$$\text{area} = \text{base} \times \text{height}.$$

For a region bounded by a curved graph, however, the height changes from point to point. The basic idea of integration is to divide the interval into small pieces, approximate the region by thin rectangles and pass to a limit.

Suppose first that f is non-negative on $[a, b]$. The area of each approximating rectangle is

$$\text{height} \times \text{width},$$

and the sum of these areas approximates the region below the graph. As the rectangles become thinner, the approximation improves.

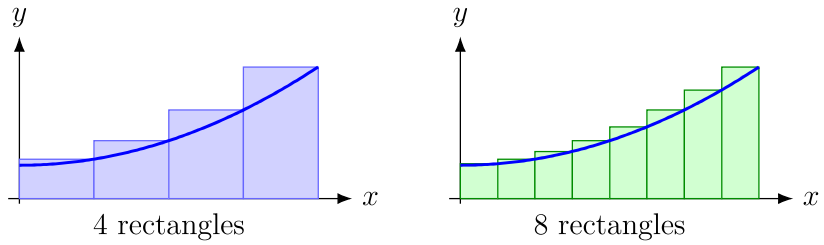


Figure 47: Using thinner rectangles improves the approximation.

Definition 20.1. A **partition** of $[a, b]$ is a finite ordered set

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

The ith subinterval is $I_i = [x_{i-1}, x_i]$, and its width is

$$\Delta x_i = x_i - x_{i-1}.$$

The **mesh** of the partition is

$$\|P\| = \max_{1 \leqslant i \leqslant n} \Delta x_i.$$

The mesh measures the width of the largest subinterval. Thus, $\|P\| \to 0$ means that every subinterval becomes short. Merely increasing the number of partition points does not guarantee this if one subinterval remains wide.

For a partition into n equal subintervals,

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x.$$

20.2 Lower and upper sums

Let $f : [a, b] \to \mathbb{R}$ be bounded. On each subinterval I_i, define

$$m_i = \inf_{x \in I_i} f(x), \qquad M_i = \sup_{x \in I_i} f(x).$$

The corresponding **lower sum** and **upper sum** are

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

If $f \geqslant 0$, the lower rectangles lie below the graph and the upper rectangles cover it. The same formulas remain valid when f takes negative values and then describe signed area.

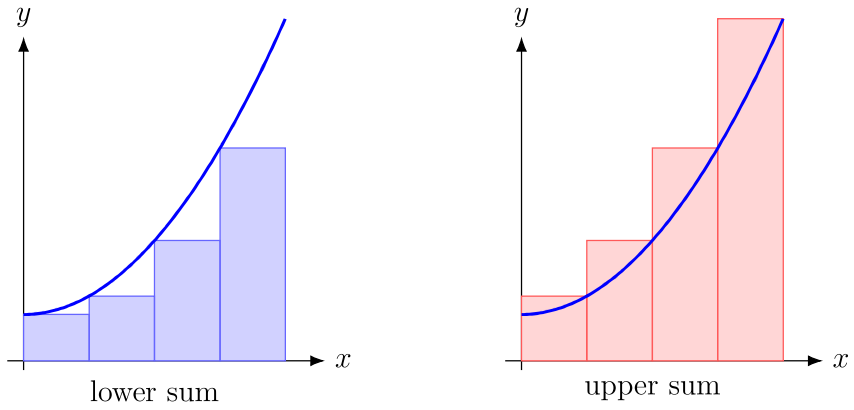


Figure 48: Lower and upper rectangular approximations for an increasing function.

A partition Q is a **refinement** of P if it contains every point of P and possibly some additional points. Refining a partition can only increase the lower sum and decrease the upper sum:

$$L(f, P) \leqslant L(f, Q) \leqslant U(f, Q) \leqslant U(f, P).$$

Thus, lower and upper sums squeeze the integral from opposite sides.

Definition 20.2. A bounded function $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** if

$$\sup_P L(f, P) = \inf_P U(f, P),$$

where the supremum and infimum are taken over all partitions of $[a, b]$. Their common value is the **Riemann integral** of f and is denoted by

$$\int_a^b f(x)\, dx.$$

The letter x is a dummy variable. For example,

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt.$$

Theorem 20.3 (Darboux criterion)**.** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if, for every $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Indeed,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i.$$

The function is integrable when its oscillation on sufficiently short subintervals makes this difference arbitrarily small.

20.3 A first integral from lower and upper sums

Example 20.4. Let $f(x) = x$ on $[0, 1]$ and divide the interval into n equal pieces:

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Since f is increasing, the left endpoints give the lower sum and the right endpoints give the upper sum. We obtain

$$L(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n} = \frac{1}{n^2} \frac{n(n-1)}{2} = \frac{n-1}{2n},$$
$$U(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}.$$

Both expressions converge to $1/2$. Hence,

$$\int_0^1 x \, dx = \frac{1}{2}.$$

This agrees with the elementary area of a right triangle whose base and height are both equal to 1.

20.4 Riemann sums

Instead of using the smallest or largest value of f on each subinterval, choose an arbitrary sample point

$$\xi_i \in [x_{i-1}, x_i].$$

The corresponding **Riemann sum** is

$$S(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

Since $m_i \leqslant f(\xi_i) \leqslant M_i$, we always have

$$L(f, P) \leqslant S(f, P, \xi) \leqslant U(f, P).$$

Theorem 20.5 (Riemann-sum characterization)**.** If f is Riemann integrable on $[a, b]$, then every sequence of Riemann sums whose meshes tend to zero converges to

$$\int_a^b f(x)\,dx.$$

The limit does not depend on the choice of sample points.

For an equal partition, $\Delta x = (b-a)/n$. Choosing the right endpoint of each subinterval gives

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n},$$

provided that f is Riemann integrable. Left endpoints, midpoints and arbitrary sample points give the same limiting value.

Example 20.6. Let us calculate $\int_0^1 x^2\,dx$ using right endpoints. We have

$$\Delta x = \frac{1}{n}, \qquad x_i = \frac{i}{n}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

Taking the limit gives

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

20.5 Which functions are integrable?

Boundedness alone does not guarantee Riemann integrability. Fortunately, most functions encountered in Calculus 1 satisfy simple sufficient conditions.

Theorem 20.7 (Useful sufficient conditions)**.** Let $f : [a, b] \to \mathbb{R}$.

- If f is continuous on $[a, b]$, then f is Riemann integrable.
- If f is bounded and monotone on $[a, b]$, then f is Riemann integrable.
- If f is bounded and has only finitely many discontinuities, then f is Riemann integrable.

Changing the value of a bounded integrable function at finitely many points does not change its integral. A finite set has no width and therefore contributes no area in the limiting sums.

Example 20.8. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & x \neq 1/2, \\ 100, & x = 1/2. \end{cases}$$

The function is discontinuous at $1/2$, but it is Riemann integrable and

$$\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

The isolated spike does not change the integral.

20.6 Properties and signed area

Proposition 20.9 (Basic properties)**.** Let f, g be Riemann integrable on $[a,b]$ and let $\alpha, \beta \in \mathbb{R}$.

- **Linearity:**
$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$
- **Additivity in the interval:** if $c \in [a,b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$
- **Order:** if $f(x) \leqslant g(x)$ for every $x \in [a,b]$, then
$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$
- **Absolute-value estimate:**
$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx.$$

If $m \leqslant f(x) \leqslant M$ on $[a,b]$, the order property gives

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

We also use the conventions

$$\int_a^a f(x)\,dx = 0, \qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Thus, the limits of integration carry an orientation.

The definite integral measures signed area. Portions of the graph below the x-axis contribute negatively. Therefore, an integral may vanish even when the function is not identically zero.

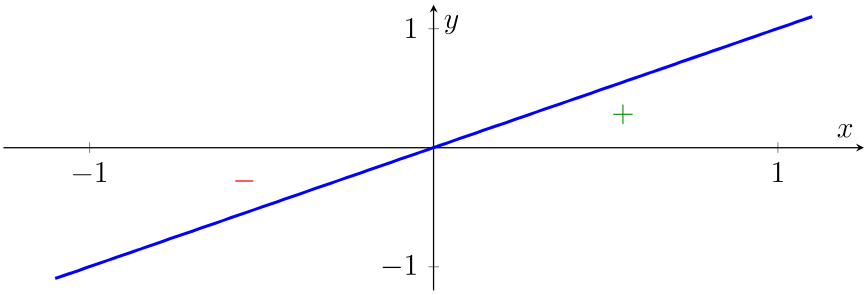


Figure 49: The two signed areas for $f(x) = x$ on $[-1, 1]$ cancel.

If f is odd and integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

If f is even, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$