21 Lecture #14: Week 8

21.1 Antiderivatives and indefinite integrals

Differentiation starts with a function and produces its rate of change. Integration also asks the reverse question: if the derivative is known, can we recover the original function?

Definition 21.1. Let $f : I \to \mathbb{R}$, where I is an interval. A function $F : I \to \mathbb{R}$ is an **antiderivative** of f on I if

$$F'(x) = f(x)$$

for every interior point $x \in I$.

For example, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$. It is not the only one, since every function $x^3 + C$ has the same derivative.

Theorem 21.2 (Antiderivatives differ by a constant)**.** If F and G are antiderivatives of the same function f on an interval I, then there exists $C \in \mathbb{R}$ such that

$$F(x) = G(x) + C$$

for every $x \in I$.

Proof. The difference $H = F - G$ satisfies

$$H' = F' - G' = f - f = 0.$$

By the Mean Value Theorem, a function whose derivative vanishes throughout an interval is constant there. Hence, $H = C$. □

The interval hypothesis matters. On the disconnected domain $\mathbb{R} \setminus \{0\}$, an antiderivative may have one constant on $(-\infty, 0)$ and another on $(0, +\infty)$.

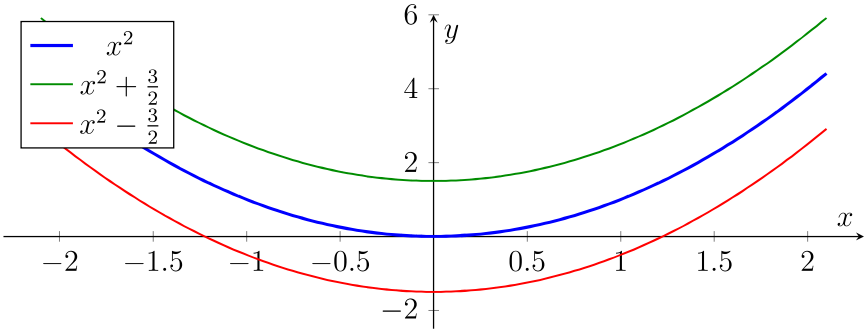


Figure 50: Every function in this family has derivative $2x$.

Definition 21.3. The **indefinite integral** of f is the family of all its antiderivatives:

$$\int f(x)\,dx = F(x) + C,$$

where $F' = f$ and $C \in \mathbb{R}$.

The symbol dx records the variable of integration. An indefinite integral is a family of functions, not an area and not a single number. The constant C must be included unless an initial condition determines it.

The following table reverses familiar derivative formulas.

Integrand	General antiderivative
x^α, $\alpha \neq -1$	$\dfrac{x^{\alpha+1}}{\alpha+1} + C$
$1/x$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
a^x, $a > 0$, $a \neq 1$	$\dfrac{a^x}{\ln a} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$1/\cos^2 x$	$\tan x + C$
$1/(1+x^2)$	$\arctan x + C$

Integration is linear. If $F' = f$ and $G' = g$, then

$$\int \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha F(x) + \beta G(x) + C.$$

Example 21.4. Calculate

$$\int (6x^2 - 4x + 3)\, dx.$$

We integrate term by term:

$$\begin{aligned}\int (6x^2 - 4x + 3)\, dx &= 6\int x^2\, dx - 4\int x\, dx + 3\int 1\, dx \\ &= 2x^3 - 2x^2 + 3x + C.\end{aligned}$$

Differentiating the answer gives the original integrand.

Example 21.5. For $x > 0$,

$$\int \left(3e^x - 2\cos x + \frac{4}{x}\right) dx = 3e^x - 2\sin x + 4\ln x + C.$$

Example 21.6. Find F if

$$F'(x) = 3x^2 - 2x, \qquad F(1) = 5.$$

First,

$$F(x) = x^3 - x^2 + C.$$

The initial condition gives

$$5 = F(1) = 1 - 1 + C,$$

so $C = 5$. Therefore,

$$F(x) = x^3 - x^2 + 5.$$

21.2 Indefinite and definite integrals

The same integral sign appears in two related but different objects. The indefinite integral

$$\int f(x)\,dx = F(x) + C$$

is a family of antiderivatives. The definite integral

$$\int_a^b f(x)\,dx$$

is one real number defined as a limit of Riemann sums. No constant C appears in a definite integral.

The properties from Lecture 13 remain available. In particular,

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g,$$
$$\int_a^b f = \int_a^c f + \int_c^b f,$$
$$\int_b^a f = -\int_a^b f.$$

If $f \leqslant g$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

21.3 Substitution

Substitution reverses the chain rule. Since

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x),$$

we have

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

Writing

$$u = g(x), \qquad du = g'(x)\,dx,$$

transforms the integral into one involving the new variable u.

The procedure is the following.

- Choose an inner expression $u = g(x)$.
- Differentiate it to obtain $du = g'(x)\,dx$.
- Rewrite the entire integral in terms of u and du.
- Integrate with respect to u.
- Return to the original variable.

- Differentiate the result to check it.

A successful substitution removes every occurrence of x from the transformed integral.

Example 21.7. Calculate

$$\int 2x\cos(x^2)\,dx.$$

The expression x^2 lies inside the cosine, and its derivative $2x$ also appears. Choose

$$u = x^2, \qquad du = 2x\,dx.$$

Then

$$\begin{aligned}\int 2x\cos(x^2)\,dx &= \int \cos u\,du \\ &= \sin u + C \\ &= \sin(x^2) + C.\end{aligned}$$

Indeed,

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2).$$

Example 21.8. Calculate

$$\int \frac{x}{\sqrt{1+x^2}}\,dx.$$

Let

$$u = 1 + x^2, \qquad du = 2x\,dx, \qquad x\,dx = \frac{1}{2}du.$$

Therefore,

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &= \frac{1}{2}\int u^{-1/2}\,du \\ &= \sqrt{u} + C \\ &= \sqrt{1+x^2} + C.\end{aligned}$$

Example 21.9. Calculate

$$\int \frac{e^x}{1+e^x}\,dx.$$

Take

$$u = 1 + e^x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int \frac{e^x}{1+e^x}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln(1+e^x) + C,\end{aligned}$$

because $1 + e^x > 0$.

One must not mix the variables u and x in the transformed integral. For example, the substitution $u = x^2$ does not directly calculate

$$\int \cos(x^2)\,dx,$$

because the necessary factor $2x$ is absent.

21.4 Integration by parts

Integration by parts reverses the product rule. From

$$(uv)' = u'v + uv'$$

we obtain

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

The method transfers a derivative from one factor to the other.

For a product, we usually choose u to be a factor that becomes simpler when differentiated and choose dv to be a factor that is easy to integrate. A useful preference for u is

$$\text{logarithmic,} \quad \text{inverse trigonometric,} \quad \text{algebraic,} \quad \text{trigonometric,} \quad \text{exponential.}$$

This is only a guide. The remaining integral determines whether the choice is useful.

Example 21.10. Calculate

$$\int xe^x\,dx.$$

Choose

$$u = x, \qquad dv = e^x\,dx.$$

Then $du = dx$ and $v = e^x$. Hence,

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C.\end{aligned}$$

Example 21.11. For

$$\int x\cos x\,dx,$$

take $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$, so

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C.\end{aligned}$$

Example 21.12. Calculate $\int \ln x\,dx$ for $x > 0$. Write $\ln x = 1 \cdot \ln x$ and choose

$$u = \ln x, \qquad dv = dx.$$

Then $du = dx/x$ and $v = x$. Therefore,

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C.\end{aligned}$$

Example 21.13. Calculate

$$\int x^2e^x\,dx.$$

The first application of integration by parts gives

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

Using the previous example,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= e^x (x^2 - 2x + 2) + C. \end{aligned}$$

The most common mistakes are forgetting the minus sign, differentiating both factors, choosing a dv that cannot be integrated, or stopping before the remaining integral has been calculated. Differentiating the final answer is the best check.

21.5 The Fundamental Theorem of Calculus

Fix a and let f be continuous. The function

$$A(x) = \int_a^x f(t) \, dt$$

records the signed area accumulated while the right endpoint moves from a to x. The variable t disappears after the integration, while x is the input of the new function.

For a small increment h,

$$A(x+h) - A(x) = \int_x^{x+h} f(t) \, dt.$$

Continuity implies that $f(t)$ is close to $f(x)$ on a short interval. Thus, the new thin region has approximate area $f(x)h$, and

$$\frac{A(x+h) - A(x)}{h} \approx f(x).$$

This observation leads to the first part of the Fundamental Theorem.

Theorem 21.14 (Fundamental Theorem of Calculus, Part I)**.** Let f be continuous on an interval I, fix $a \in I$, and define

$$A(x) = \int_a^x f(t) \, dt.$$

Then A is differentiable on the interior of I and

$$A'(x) = f(x).$$

The derivative of accumulated area is the current height of the function. This theorem constructs an antiderivative even when no elementary formula for it is available.

Example 21.15. Let

$$F(x) = \int_1^x \sqrt{1 + t^4} \, dt.$$

The integrand is continuous, so

$$F'(x) = \sqrt{1 + x^4}.$$

There is no need to calculate the integral first.

Example 21.16. Let

$$G(x) = \int_0^x e^{-t^2}\,dt.$$

Although e^{-t^2} has no elementary antiderivative,

$$G'(x) = e^{-x^2}.$$

If the upper endpoint depends on x, the chain rule must also be used.

Example 21.17. Let

$$H(x) = \int_0^{x^2} \cos(t^3)\,dt.$$

Writing

$$A(u) = \int_0^u \cos(t^3)\,dt,$$

we have $H(x) = A(x^2)$ and $A'(u) = \cos(u^3)$. Therefore,

$$H'(x) = A'(x^2)\,2x = 2x\cos(x^6).$$

Theorem 21.18 (Derivative with variable endpoints)**.** Let f be continuous and let u and v be differentiable. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x).$$

The minus sign belongs to the lower endpoint.

Example 21.19. If

$$K(x) = \int_x^{x^2} e^{t^2}\,dt,$$

then

$$K'(x) = 2xe^{x^4} - e^{x^2}.$$

The second part of the Fundamental Theorem turns definite integration into the evaluation of an antiderivative.

Theorem 21.20 (Fundamental Theorem of Calculus, Part II)**.** Let f be continuous on $[a, b]$, and let F be any antiderivative of f on that interval. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We often write

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b.$$

Proof. Define

$$A(x) = \int_a^x f(t)\,dt.$$

By Part I, $A' = f$. Since $F' = f$ as well, the difference $A - F$ is constant. At $x = a$, $A(a) = 0$, and therefore

$$A(x) = F(x) - F(a).$$

Taking $x = b$ gives the result. □

The constant of integration cancels in $F(b) - F(a)$, so it is not written when evaluating a definite integral.

Example 21.21. Calculate

$$\int_0^{\pi/2} \cos x \, dx.$$

Since $\sin x$ is an antiderivative of $\cos x$,

$$\begin{aligned}\int_0^{\pi/2} \cos x \, dx &= [\sin x]_0^{\pi/2} \\ &= \sin \frac{\pi}{2} - \sin 0 \\ &= 1.\end{aligned}$$

Substitution may also be applied directly to a definite integral. In this case, the limits must be changed to the new variable.

Example 21.22. Calculate

$$\int_0^1 2xe^{x^2} \, dx.$$

Let $u = x^2$, so $du = 2x \, dx$. The new limits are

$$x = 0 \Longrightarrow u = 0, \qquad x = 1 \Longrightarrow u = 1.$$

Therefore,

$$\begin{aligned}\int_0^1 2xe^{x^2} \, dx &= \int_0^1 e^u \, du \\ &= [e^u]_0^1 \\ &= e - 1.\end{aligned}$$

For definite integrals, integration by parts takes the form

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du.$$

Example 21.23. Calculate

$$\int_0^1 xe^x \, dx.$$

Take $u = x$ and $dv = e^x \, dx$. Then

$$\begin{aligned}\int_0^1 xe^x \, dx &= [xe^x]_0^1 - \int_0^1 e^x \, dx \\ &= e - [e^x]_0^1 \\ &= e - (e - 1) \\ &= 1.\end{aligned}$$

The two parts of the Fundamental Theorem show that differentiation and integration undo each other. Part I creates an antiderivative from an accumulation function, while Part II uses an antiderivative to calculate the accumulated value.